Vacuum Branes in D-Dimensional Static Spacetimes with Spatial Symmetry $IO(D-2)$, $O(D-1)$ or $O_+(D-2,1)$

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In this paper, we give a complete classification of vacuum branes, i.e., everywhere umbilical time-like hypersurfaces whose extrinsic curvature is a constant multiple of the induced metric, $K_{\mu\nu} = \sigma g_{\mu\nu}$, in D-dimensional static spacetimes with spatial symmetry $G(D-2,K)$, where $G(n,K)$ is the isometry group of an n-dimensional space with constant sectional curvature K. $D \geq 4$ is assumed. It is shown that all possible configurations of a brane are invariant under an isometry subgroup $G(D-3,K')$ for some $K' \geq K$. In particular, configurations of a brane with $\sigma \neq 0$ are always $G(D-2,K)$ invariant, except for those in five special one-parameter families of spacetimes. Further, such $G(D-2,K)$-invariant configurations are allowed only in spacetimes whose Ricci tensors are isotropic in the two planes orthogonal to each $G(D-2,K)$-orbit, or for special values of σ, which do not exist in generic cases. On the basis of these results, we prove the non-existence of a vacuum brane with black hole geometry in static bulk spacetimes with spatial symmetry $G(D-2,K)$. We also discuss mathematical implications of these results.

§1. Introduction

Recently, braneworld models in which our universe is realized as a boundary or a subspace, called a brane, of a higher-dimensional spacetime have been actively studied as possible new universe models based on higher-dimensional unified theories. For example, in the models proposed by Randall and Sundrum,[1),2)] our universe is modeled as a boundary of a 5-dimensional vacuum spacetime with negative cosmological constant whose extrinsic curvature $K_{\mu\nu}$ is related to the energy-momentum tensor $T_{\mu\nu}$ of the universe by $\kappa_5^2 T_{\mu\nu} = 2(K^\mu_\nu - K^\lambda_\lambda \delta^\mu_\nu)$, where κ_5^2 is the gravitational coupling constant in a 5-dimensional spacetime called a bulk spacetime. In this model, when the universe is empty, the extrinsic curvature becomes a constant multiple of the induced metric $g_{\mu\nu}$:

$$K_{\mu\nu} = \sigma g_{\mu\nu}. \tag{1·1}$$

From the Gauss equation, the brane geometry becomes Minkowskian only for some non-zero value of σ determined by the cosmological constant when the bulk spacetime is an anti-de Sitter spacetime. In more realistic models, the bulk spacetime is not a vacuum and contains various fields, such as a radion stabilizer and moduli fields.[3)–6)] Even in these models, the extrinsic curvature of the brane satisfies the condition (1·1), provided that the universe is empty and a $\mathbb{Z}_2$ symmetry is imposed at the brane.

The viability of the braneworld scenario has been investigated with respect to various aspects, such as the behavior of local gravity and gravitational waves, the existence and the behavior of FRW universe models, and the behavior of cosmological

perturbations.[2),7)–19)] However, it is still unknown what kind of black holes will be produced by the gravitational collapse of stars. A natural starting point for the investigation of this problem is to find a regular solution that represents a black hole in the vacuum brane and has a localized horizon in the bulk spacetime, but no such solution has yet been found, although various suggestive arguments have been presented.[20)–22)] From the uniqueness theorems for black holes in 4-dimensional spacetimes (for reviews, see Refs. 23)–26)) and that for static black holes in higher dimensions,[27)] one may naively expect that the Schwarzschild-anti-de Sitter solution provides such a solution. However, it does not give a black hole solution in the brane, because this solution does not contain a non-spherically symmetric vacuum brane, as pointed out by Chamblin, Hawking and Reall.[20)] This result is consistent with the analysis based on the C-metric solution in the 4-dimensional bulk spacetime.[21),22)]

One purpose of the present paper is to generalize this result by Chamblin, Hawking and Reall and to show that there exists no vacuum brane configuration with black hole geometry in a D-dimensional static spacetime with spatial symmetry $G(D-2, K)$, where $G(n, K)$ is the isometry group of an n-dimensional space with constant sectional curvature K. We assume that $D \geq 4$, but do not impose the Einstein equations on the metric of the bulk spacetime, in order to make the results applicable to more general braneworld models. We also show that the existence of a vacuum brane strongly constrains the geometry of the bulk spacetime.

The second purpose of this paper is rather mathematical. In mathematical terminology, a vacuum brane with $\sigma = 0$ represents a totally geodesic hypersurface. Hence, our analysis completely determines all possible totally geodesic time-like hypersurfaces in static spacetimes with the assumed spatial symmetry. In contrast, a vacuum brane with $\sigma \neq 0$ is only geodesic with respect to null geodesics. However, such a hypersurface belongs to a more general class called 'everywhere umbilical hypersurfaces', which are defined as hypersurfaces whose extrinsic curvature is proportional to the induced metric.[*)] Concerning such hypersurfaces in Euclidean spaces, there is a well-know theorem that an everywhere umbilical hypersurface is a hyperplane or a hypersphere.[28)] We extend this theorem to everywhere umbilical surfaces in non-flat spacetimes including constant curvature spacetimes. For this purpose, we do assume no symmetry property on a brane initially, and instead determine what kind of symmetry the brane should have. We also extend the well-known rigidity theorem on hypersurfaces in Euclidean spaces to non-flat spacetimes.

The paper is organized as follows. In the next section, we give a general formula expressing the extrinsic curvature of a hypersurface in a generic spacetime in terms of a coordinate relation specifying the hypersurface, and derive basic equations describing a vacuum brane in the class of spacetimes considered in the present paper. Then, in §§3 and 4, we solve these basic equations to find all possible solutions for the brane configuration as well as constraints on the geometry of the bulk spacetime. In §5, we classify isometry classes of the solutions found in these two sections and

[*)] The proportionality coefficient σ for an everywhere umbilical hypersurface need not be constant on the hypersurface in general. However, it is constrained to be constant by the Codazzi equation when the embedding bulk space(-time) has a constant curvature, as shown in §5.

clarify their geometrical features. Finally, §6 is devoted to a summary and discussion. Some geometrical formulas and proofs of auxiliary theorems used in the text are given in Appendices.

§2. Basic equations

In this section, we derive the basic equations that determine possible configurations of a vacuum brane in a static bulk of dimension $D(\geq 4)$ with spatial symmetry $O(D-1)$, $O_+(D-2,1)$ or $IO(D-2)$.

2.1. *Extrinsic curvature of a hypersurface*

We first derive a general expression for the extrinsic curvature of a hypersurface Σ in a D-dimensional general spacetime,

$$ds_D^2 = \tilde{g}_{MN} dx^M dx^N. \tag{2·1}$$

(In this section, capital Latin indices $L, M, \cdots, R$ run over $0, \cdots, D-1$.) Let y^M be another coordinate system such that $y^{D-1} = 0$ on the hypersurface Σ and y^μ ($\mu = 0, 1, \cdots, D-2$) gives an intrinsic coordinate system of Σ. Then, the Christoffel symbol $\tilde{\Gamma}'^L_{MN}$ in the y-coordinates is related to the Christoffel symbol $\tilde{\Gamma}^L{}_{MN}$ in the x-coordinates by

$$\frac{\partial x^L}{\partial y^P} \tilde{\Gamma}'^P_{QR} = \tilde{\Gamma}^L_{MN} \frac{\partial x^M}{\partial y^Q} \frac{\partial x^N}{\partial y^R} + \frac{\partial^2 x^L}{\partial y^Q \partial y^R}. \tag{2·2}$$

Now, let n^M be the components of the unit normal to Σ in the x-coordinates and n'^M be those in the y-coordinates. Then, from this relation and the coordinate transformation law of the normal vector, the extrinsic curvature of Σ in the y-coordinates is expressed as

$$\begin{aligned} K_{\mu\nu} &= -\tilde{\nabla}_\mu n'_\nu = \tilde{\Gamma}'^M_{\mu\nu} n'_M \\ &= n_L \left(\tilde{\Gamma}^L_{MN} \partial_\mu x^M \partial_\nu x^N + \partial_\mu \partial_\nu x^L \right), \end{aligned} \tag{2·3}$$

where ∂_μ is the derivative with respect to the y^μ-coordinate, and it is understood that the value on Σ is taken on the right-hand side of this equation.*) Further, the components of the normal vector are expressed as

$$n_L = \pm \tilde{N}^{-1} \epsilon_{L M_1 \cdots M_{D-1}} \frac{\partial x^{M_1}}{\partial y^0} \cdots \frac{\partial x^{M_{D-1}}}{\partial y^{D-2}}, \tag{2·4}$$

where $\tilde{N}$ is a positive normalization constant determined by the condition $\tilde{g}^{MN} n_M n_N = 1$.

In subsequent applications, we specify the hypersurface by picking out one coordinate, say x^{D-1}, and expressing x^{D-1} as a function of the other coordinates,

*) Note that, in spite of its non-tensorial appearance, the right-hand side of (2·3) is invariant under an arbitrary change of the coordinates x^M and transforms as a 2nd-rank tensor on Σ under a change of the y^μ-coordinates.

$X(x^0, \cdots, x^{D-2})$. In this case, it is natural to choose the y-coordinates such that $x^\mu = y^\mu$ for $\mu = 0, \cdots, D-2$ and $x^{D-1} = y^{D-1} + X(y^0, \cdots, y^{D-2})$. Then, by taking the direction of n^M so that $n_{D-1} > 0$, the expression for n_L is simplified as

$$n_{D-1} = \tilde{N}^{-1}, \quad n_\mu = -\tilde{N}^{-1}\partial_\mu X, \tag{2·5}$$

and $\tilde{N}$ is given by

$$\tilde{N}^2 = \tilde{g}^{D-1\,D-1} - 2\tilde{g}^{\mu\,D-1}\partial_\mu X + \tilde{g}^{\mu\nu}\partial_\mu X \partial_\nu X. \tag{2·6}$$

Further, (2·3) reduces to

$$\begin{aligned}\tilde{N}K_{\mu\nu} = \partial_\mu\partial_\nu X + \tilde{\Gamma}^{D-1}_{\mu\nu} - \partial_\lambda X \tilde{\Gamma}^\lambda_{\mu\nu} + 2\tilde{\Gamma}^{D-1}_{D-1(\mu}\partial_{\nu)}X - 2\partial_\lambda X \tilde{\Gamma}^\lambda_{D-1(\mu}\partial_{\nu)}X \\ +(\tilde{\Gamma}^{D-1}_{D-1\,D-1} - \partial_\lambda X \tilde{\Gamma}^\lambda_{D-1\,D-1})\partial_\mu X \partial_\nu X.\end{aligned} \tag{2·7}$$

Similarly, the induced metric $g_{\mu\nu}$ of Σ is expressed in terms of $\tilde{g}_{MN}$ and X as

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + 2\tilde{g}_{D-1(\mu}\partial_{\nu)}X + \tilde{g}_{D-1\,D-1}\partial_\mu X \partial_\nu X. \tag{2·8}$$

2.2. *Equation for the brane configuration*

Now, we apply the formulas derived above to a vacuum brane in a D-dimensional static bulk spacetime with spatial symmetry $G(D-2, K)$, whose metric is given by

$$ds_D^2 = -U(R)dT^2 + \frac{dR^2}{W(R)} + S(R)^2 d\sigma_{D-2}^2; \tag{2·9}$$

$$d\sigma_{D-2}^2 = \gamma_{ij}dz^i dz^j = d\chi^2 + \rho(\chi)^2 d\Omega_{D-3}^2, \tag{2·10}$$

where $d\sigma_{D-2}^2$ is the metric of a $(D-2)$-dimensional constant curvature space M_K^{D-2} with sectional curvature K $(K = 0, \pm 1)$, $d\Omega_{D-3}^2 = \hat{\gamma}_{AB}d\theta^A d\theta^B$ is the metric of the $(D-3)$-dimensional unit sphere, and $\rho(\chi)$ is given by

$$\rho(\chi) = \begin{cases} \sin\chi; & O(D-1)\text{-symmetric } (K=1), \\ \sinh\chi; & O_+(D-2,1)\text{-symmetric } (K=-1), \\ \chi; & IO(D-2)\text{-symmetric } (K=0). \end{cases} \tag{2·11}$$

Note that the expression (2·9) has a gauge freedom corresponding to an arbitrary reparametrization of the R-coordinate. We fix this gauge freedom by setting $S(R) = R$ when $S' \not\equiv 0$, and $W(R) = 1$ and $S(R) = S_0$=const when $S' \equiv 0$. We do not impose the Einstein equations on the bulk geometry in the present paper. Therefore, at this point, $W(R)$ in the case $S(R) = R$ and $U(R)$ are arbitrary functions that are positive for some range of R.

In the present paper, we only consider a vacuum brane of the Randall-Sundrum type. Hence, the brane configuration is determined by the condition (1·1). In general, the brane configuration can be represented as a level surface of some function F on the spacetime as

$$\Sigma : F(T, R, z^i) = 0. \tag{2·12}$$

Since a brane is a time-like hypersurface, we can assume that $\partial_R F \not\equiv 0$ or $\partial_i F \not\equiv 0$. In the first case, the brane configuration can be written $R = R(T, \boldsymbol{z})$, and from the general formulas in the previous subsection and the formulas for the Christoffel symbols in Appendix A, (1·1) gives the following equations for $R(T, \boldsymbol{z})$:

$$R_{TT} - \left(\frac{W'}{2W} + \frac{U'}{U}\right) R_T^2 + \frac{1}{2} W U' = \sigma N \left(-U + \frac{R_T^2}{W}\right), \tag{2·13a}$$

$$R_{Ti} - \left(\frac{S'}{S} + \frac{U'}{2U} + \frac{W'}{2W}\right) R_T R_i = \sigma N \frac{R_T R_i}{W}, \tag{2·13b}$$

$$D_i D_j R - \left(\frac{2S'}{S} + \frac{W'}{2W}\right) R_i R_j - S S' W \gamma_{ij} = \sigma N \left(\frac{R_i R_j}{W} + S^2 \gamma_{ij}\right), \tag{2·13c}$$

$$N^2 = W - \frac{R_T^2}{U} + \frac{R_i R_j \gamma^{ij}}{S^2}. \tag{2·13d}$$

Here, the subscripts T and i on R represent differentiations with respect to T and z^i, respectively, D_i is the covariant derivative with respect to γ_{ij} on M_K^{D-2}, and the prime denotes differentiation with respect to the argument of the corresponding function.

In the case $\partial_i F \not\equiv 0$, the brane configuration can be written $\chi = \chi(T, R, \boldsymbol{\theta})$ with an appropriate choice of the coordinate system of the $(D-2)$-dimensional constant curvature space. For this choice, (1·1) gives the following equations for $\chi(T, R, \boldsymbol{\theta})$:

$$\chi_{TT} + S S' W \chi_R \chi_T^2 - \frac{W U'}{2} \chi_R = \sigma N \left(-\frac{U}{S} + S \chi_T^2\right), \tag{2·14a}$$

$$\chi_{TR} + \left(-\frac{U'}{2U} + \frac{S'}{S}\right) \chi_T + S S' W \chi_R^2 \chi_T = \sigma N S \chi_T \chi_R, \tag{2·14b}$$

$$\chi_{RR} + \left(\frac{W'}{2W} + \frac{2S'}{S}\right) \chi_R + S S' W \chi_R^3 = \sigma N \left(\frac{1}{SW} + S \chi_R^2\right), \tag{2·14c}$$

$$\chi_{TA} - \frac{\rho'}{\rho} \chi_T \chi_A + S S' W \chi_R \chi_T \chi_A = \sigma N S \chi_T \chi_A, \tag{2·14d}$$

$$\chi_{RA} - \frac{\rho'}{\rho} \chi_R \chi_A + S S' W \chi_R^2 \chi_A = \sigma N S \chi_R \chi_A, \tag{2·14e}$$

$$\begin{aligned} &\hat{D}_A \hat{D}_B \chi - 2\frac{\rho'}{\rho} \chi_A \chi_B + S S' W \chi_A \chi_B \chi_R + (S S' W \rho^2 \chi_R - \rho \rho') \hat{\gamma}_{AB} \\ &\quad = N \sigma S (\chi_A \chi_B + \rho^2 \hat{\gamma}_{AB}), \end{aligned} \tag{2·14f}$$

$$N^2 = 1 - \frac{S^2 \chi_T^2}{U} + S^2 W \chi_R^2 + \frac{1}{\rho^2} \chi_A \chi_B \hat{\gamma}^{AB}. \tag{2·14g}$$

Here, the subscripts T, R and A for χ represent the derivatives with respect to T, R and θ^A, respectively, and $\hat{D}_A$ is the covariant derivative with respect to the metric $\hat{\gamma}_{AB}$ on the unit sphere S^{D-3}.

§3. Solutions in spacetimes with $S = R$

In this section, we solve the equations for the brane configuration derived in the previous section for the bulk metric with $S(R) = R$. We divide the problem into the case in which the brane configuration has non-trivial R-dependence and the case in which the brane configuration is independent of R. The former case is described by (2·13), while the latter case is determined by (2·14) with $\chi = \chi(T, \boldsymbol{\theta})$.

3.1. $R = R(T, \boldsymbol{z})$-*type configurations*

In order to solve (2·13), we replace the variable R by the new variable $r = r(R)$ defined by

$$dr = \frac{dR}{R\sqrt{W}} \tag{3·1}$$

and introduce $V(R)$ and $Z(R)$ defined by

$$V = \frac{\sqrt{U}}{R}, \quad Z = \frac{\sqrt{W}}{R}. \tag{3·2}$$

In terms of these variables, (2·13b) is written

$$r_{Ti} = \left(\frac{V_r}{V} + RZ + \frac{\sigma N}{Z}\right) r_T r_i. \tag{3·3}$$

As shown in Appendix B, this equation implies that r depends on $\boldsymbol{z} = (z^i)$ through some function $X(\boldsymbol{z})$ as $r = r(T, X(\boldsymbol{z}))$. Hence, from this point, we regard r as a function of the two variables T and X. Then, (2·13) can be written

$$r_{TT} + \frac{V_r}{V}(V^2 - 2r_T^2) + \left(RZ + \frac{\sigma N}{Z}\right)(V^2 - r_T^2) = 0, \tag{3·4a}$$

$$r_{TX} = \left(\frac{V_r}{V} + RZ + \frac{\sigma N}{Z}\right) r_T r_X, \tag{3·4b}$$

$$r_X D_i D_j X + \left[r_{XX} - r_X^2\left(RZ + \frac{\sigma N}{Z}\right)\right] X_i X_j = \left(RZ + \frac{\sigma N}{Z}\right)\gamma_{ij}, \tag{3·4c}$$

$$\frac{N^2}{R^2 Z^2} = 1 - \frac{r_T^2}{V^2} + r_X^2 (DX)^2, \tag{3·4d}$$

where $(DX)^2 = X_i X_j \gamma^{ij}$.

3.1.1. Configurations represented as $R = R(T)$

First, we treat the special case in which $r_i \equiv 0$, i.e., $r_X \equiv 0$. In this case, (3·4c) gives $\sigma N = -RZ^2$, which is equivalent to $\sigma < 0$ and

$$\frac{1}{Z^2}\left(\frac{r_T^2}{V^2} - 1\right) = -\frac{1}{\sigma^2}. \tag{3·5}$$

Differentiation of this equation with respect to T gives

$$r_T\left[r_{TT} + \frac{V_r}{V}(V^2 - 2r_T^2) + (r_T^2 - V^2)\left(\frac{V_r}{V} - \frac{Z_r}{Z}\right)\right] = 0. \tag{3·6}$$

Furthermore, in the present case, (3·4a) reads

$$r_{TT} + \frac{V_r}{V}(V^2 - 2r_T^2) = 0. \tag{3·7}$$

Comparing these two equations, we find that $V_r/V = Z_r/Z$ for $r_T \neq 0$. Hence, U is proportional to W in the range of R in which $R_T \neq 0$. Since U can be multiplied by an arbitrary positive constant by rescaling the time variable T, this implies that we can set $U = W$. Note that this condition is equivalent to the condition that R^a_b is proportional to δ^a_b for $a, b = R, T$. Under this condition, the brane configuration equations are equivalent to the condition $\sigma < 0$ and the single equation (3·5), which is expressed in terms of the original variable as

$$R_T^2 = U^2 \left(1 - \frac{U}{\sigma^2 R^2}\right). \tag{3·8}$$

This type of brane configuration represents a FRW-type cosmological model. On the other hand, if $r_T \equiv 0$ (i.e., the configuration is represented by R =const), (3·5) and (3·7) give the two conditions

$$\frac{U'}{U} = \frac{2}{R}, \quad \sigma^2 = \frac{W}{R^2}. \tag{3·9}$$

Although no general constraint on U or W arises in this case, values of R satisfying these equations, if they exist, form a discrete set when U is not proportional to R^2 in any finite range of R, and the value of σ also becomes discrete.

3.1.2. Configurations with $R_i \not\equiv 0$

Next, we consider the case in which $r_i \not\equiv 0$, i.e., $r_X \not\equiv 0$ and $X_i \not\equiv 0$. In this case, (3·4c) is written

$$D_i D_j X = \alpha \gamma_{ij} + \beta D_i X D_j X, \tag{3·10}$$

where

$$\alpha = \frac{1}{r_X}\left(RZ + \frac{\sigma N}{Z}\right), \quad \beta = -\frac{r_{XX}}{r_X} + r_X^2 \alpha. \tag{3·11}$$

Since r is a function of T and X, it follows from (3·4b) that α can also be written $\alpha = \alpha(T, X(\boldsymbol{z}))$. Hence, β also depends on $\boldsymbol{z} = (z^i)$ only through $X(\boldsymbol{z})$. Further, since X is independent of T, and since $X_i X_j$ is linearly independent of γ_{ij} under the condition $X_i \not\equiv 0$, α and β should be T-independent. Therefore, (3·10) is an equation of the type discussed in Appendix C. As shown there, by replacing X by some monotonic function of X, we can always make β vanish. Since we have the freedom to make such a redefinition of X in the present case, we can assume that X satisfies the condition $\beta = 0$. Then, from the argument in Appendix C, α must be of the form $-KX + \alpha_0$, where α_0 is a constant. Thus, (3·4c) is equivalent to the set of equations

$$D_i D_j X = (\alpha_0 - KX)\gamma_{ij}, \tag{3·12a}$$

$$RZ + \frac{\sigma N}{Z} = (\alpha_0 - KX) r_X, \tag{3·12b}$$

$$r_{XX} = r_X^3 (\alpha_0 - KX). \tag{3·12c}$$

Now, we show that the consistency of the equations for the brane configuration requires W to have the form

$$W = K - \lambda R^2, \tag{3·13}$$

where λ is a constant. First, from (3·12b) and (3·12c), (3·4b) can be written

$$r_{TX} = \left(\frac{V_r r_X}{V} + \frac{r_{XX}}{r_X} \right) r_T. \tag{3·14}$$

This implies that $r_T / V r_X$ is independent of X, and hence r_T can be written

$$r_T = \dot{g}(T) V r_X, \tag{3·15}$$

in terms of some function $g(T)$. Next, as shown in Appendix C, $(DX)^2$ is written

$$(DX)^2 = c - KX^2 + 2\alpha_0 X, \tag{3·16}$$

where c is a constant. Further, the integration of (3·12c) with respect to X yields

$$-\frac{1}{r_X^2} = 2\alpha_0 X - KX^2 - f(T)^2 - \dot{g}(T)^2 + c, \tag{3·17}$$

where $f(T)^2$ is an arbitrary function of T. Inserting these expressions into (3·4d), we obtain

$$\frac{N^2}{R^2 Z^2} = f^2 r_X^2, \tag{3·18}$$

which implies that f can be taken to be positive definite. Hence, (3·12b) can be written

$$\frac{\alpha_0 - KX}{R} - \frac{Z}{r_X} = \sigma \frac{r_X}{|r_X|} f = \pm \sigma f. \tag{3·19}$$

Since the right-hand of this equation is independent of X, differentiation of this equation with respect to X yields

$$Z_r = -\frac{K}{R}. \tag{3·20}$$

This equation with $(1/R)_r = -Z$ derived from the definition of r leads to $(Z^2 - K/R^2)_r = 0$, which implies that $(W - K)/R^2$ is constant.

Up to this point, we have not used (3·4a). Next, we show that the consistency of this equation with the others provides additional strong constraints. With the help of (3·12b) and (3·12c), (3·4a) can be rewritten

$$r_{TT} - \left(\frac{2V_r}{V} + \frac{r_{XX}}{r_X^2} \right) r_T^2 + V^2 \left(\frac{V_r}{V} + \frac{r_{XX}}{r_X^2} \right) = 0. \tag{3·21}$$

Since the sum of the first two terms on the left-hand side of this equation is equal to $Vr_X(r_T/Vr_X)_T$, using (3·15), it can be further rewritten as

$$\ddot{g} + \frac{V_r}{r_X} + (\alpha_0 - KX)V = 0. \tag{3·22}$$

Also, differentiating (3·17) with respect to T and eliminating r_{TX} using (3·15), we obtain

$$\dot{g}\left[\ddot{g} + \frac{V_r}{r_X} + (\alpha_0 - KX)V\right] = -f\dot{f}. \tag{3·23}$$

Thus, the consistency of these two equations requires that f be constant. Then, differentiation of (3·19) with respect to T gives

$$0 = \pm\sigma\dot{f} = (-VZ_r + V_r Z)\dot{g}. \tag{3·24}$$

From this, it follows that if $\dot{g} \not\equiv 0$, then $V_r/V = Z_r/Z$, which implies that U is proportional to W. Thus, we find that the equations for the brane configuration have a solution $R = R(T, \boldsymbol{z})$ such that R_T and R_i do not vanish identically only when $U = W = K - \lambda R^2$ up to a constant rescaling of T, i.e., only when the bulk spacetime is either a de Sitter spacetime dS^D ($\lambda > 0$), an anti-de Sitter spacetime AdS^D ($\lambda < 0$), or a Minkowski spacetime $E^{D-1,1}$.

To summarize the argument up to this point, the original equations for the brane configuration (2·13) can be replaced by (3·15), (3·17), (3·19), (3·22) and $W = K - \lambda R^2$. The integrability condition for the first two equations with respect to r is given by the condition that f is a positive constant, which will be denoted by a from this point. Further, the third is consistent with the others only if $\dot{g} \equiv 0$ or $U = W = K - \lambda R^2$ up to a constant rescaling of T. Finally, from the argument in Appendix C, $X = X(\boldsymbol{z})$ is determined by (3·12a) uniquely up to an isometry of the constant curvature space M_K^{D-2} and multiplication by a constant that is uniquely determined by c in (3·16), except in the case $K = -1$ and $c = 0$.

In order to proceed further, we must consider the cases $R_T \not\equiv 0$ and $R_T \equiv 0$ separately.

(A) Solutions with $R_T \not\equiv 0$ and $R_i \not\equiv 0$

In this case, we can assume that $U = W = K - \lambda R^2$ without loss of generality. We consider the three cases $K = 0, \pm 1$ separately.

(A-1) The case $K = 0$

In this case, $\lambda < 0$ and $V = Z = \sqrt{-\lambda}$. Hence, r is related to R as

$$\sqrt{-\lambda} r = -\frac{1}{R}, \tag{3·25}$$

and the equations for the brane configuration are written

$$r_T = \sqrt{-\lambda}\dot{g} r_X, \tag{3·26a}$$

$$\frac{1}{r_X^2} = -2\alpha_0 X + a^2 + \dot{g}^2 - c, \tag{3·26b}$$

$$-\frac{1}{r_X}=\alpha_0 r\pm\frac{a\sigma}{\sqrt{-\lambda}}, \tag{3·26c}$$

$$\ddot{g}+\alpha_0\sqrt{-\lambda}=0. \tag{3·26d}$$

As shown in Appendix C, if we choose the Cartesian coordinate system of $M_0^{D-2}=E^{D-2}$ as z^i, X is a linear function of z^i for $\alpha_0=0$, while it can be chosen to be $(\boldsymbol{z}-\boldsymbol{z_0})^2$ for $\alpha_0\neq 0$, where $\boldsymbol{z_0}$ is a constant vector. Hence, these two cases are qualitatively different.

i) The case $\alpha_0=0$: First, in the case $\alpha_0=0$, we can normalize X so that $c=(DX)^2=1$. Since $\pm$ on the right-hand side of (3·26c) coincides with the sign of r_X, it follows that $\sigma<0$. Equations (3·26b) and (3·26c) can be easily integrated, yielding

$$r=\mp\frac{\sqrt{-\lambda}}{\sigma a}X+h(T), \tag{3·27}$$

$$\dot{g}^2=1-\left(1+\frac{\sigma^2}{\lambda}\right)a^2. \tag{3·28}$$

Inserting the first equation into (3·26a), we obtain

$$h=\pm\frac{\lambda}{\sigma a}g, \tag{3·29}$$

where we have absorbed an integration constant into g. Finally, (3·26d) determines g to be a linear function of T. Thus, the general solution for the case $K=\alpha_0=0$ is written

$$\begin{aligned}&\frac{1}{R}=\frac{\lambda}{\sigma}(\boldsymbol{p}\cdot\boldsymbol{z}+qT)+A;\\&\lambda\boldsymbol{p}^2+q^2=\sigma^2+\lambda,\ \sigma<0,\end{aligned} \tag{3·30}$$

where $\boldsymbol{z}=(z^i)$ is the Cartesian coordinate system for E^{D-2}, and $\boldsymbol{p}=(p_i)$ is a constant vector.

ii) The case $\alpha_0\neq 0$: In this case, we can set $X=(\boldsymbol{z}-\boldsymbol{z}_0)^2$ without loss of generality, i.e., $c=0$ and $\alpha_0=2$. Then, (3·26d) becomes $\ddot{g}+2\sqrt{-\lambda}=0$, whose general solution is

$$g=-\sqrt{-\lambda}(T-T_0)^2+b, \tag{3·31}$$

where T_0 and b are constants. From this expression, (3·26b) and (3·26c), we obtain

$$\left(\frac{1}{R}\mp\frac{a\sigma}{2}\right)^2=\lambda(\boldsymbol{z}-\boldsymbol{z}_0)^2+\lambda^2(T-T_0)^2-\frac{1}{4}\lambda a^2. \tag{3·32}$$

It can be easily checked that this satisfies (3·26a) and (3·26b). Hence, this is the general solution for the case $K=0$ and $\alpha_0\neq 0$.

(A-2) The case $K=\pm 1$

In this case, we can set $\alpha_0 = 0$ by a constant shift of X. Then, the equations for the brane configuration are written

$$r_T = \dot{g}(T) V r_X, \tag{3·33a}$$

$$\frac{1}{r_X^2} = KX^2 + a^2 + \dot{g}^2 - c, \tag{3·33b}$$

$$\frac{KX}{R} + \frac{V}{r_X} = \mp\sigma a, \tag{3·33c}$$

$$\ddot{g} + \frac{V_r}{r_X} - KXV = 0. \tag{3·33d}$$

From the first equation, it follows that

$$(X + gV)_T = \left(\frac{1}{r_X} + gV_r\right) r_T. \tag{3·34}$$

Also, we have the identity

$$(X + gV)_X = \left(\frac{1}{r_X} + gV_r\right) r_X. \tag{3·35}$$

The consistency of these two equations requires that $X + gV$ depend only on r (or equivalently on R). Therefore, we write

$$X + gV = h(r), \quad \frac{1}{r_X} + gV_r = h_r, \tag{3·36}$$

where the second equation automatically follows from the first. Hence, (3·33a) is equivalent to the first equation. Further, with the help of these equations and $V_r = -K/R$, (3·33c) is deformed to

$$\frac{K}{R} h + V h_r = \mp a\sigma, \tag{3·37}$$

which provides an equation to determine $h(r)$.
i) The case $\lambda \neq 0$: First, for the case $\lambda \neq 0$, since $(1/R)_r = -Z = -V$ and $V_r = Z_r = -K/R$, $h = \mp\frac{a\sigma}{\lambda R}$ gives a special solution to (3·37) and the general solution is expressed as the sum of this special solution and a constant multiple of V. The latter can be absorbed into the definition of g, which has the freedom to add an arbibrary constant. Hence, X is expressed as

$$X = -gV \mp \frac{a\sigma}{\lambda R}, \tag{3·38}$$

which in turn gives

$$\frac{1}{r_X} = g\frac{K}{R} \pm \frac{a\sigma}{\lambda} V. \tag{3·39}$$

Inserting this expression into (3·33b) gives

$$\dot{g}^2 - K\lambda g^2 = c - \left(1 + \frac{\sigma^2}{\lambda}\right) a^2. \tag{3·40}$$

Further, (3·33d) reads

$$\ddot{g} - K\lambda g = 0. \tag{3·41}$$

This equation is obtained from (3·40) by differentiating it with respect to T, if $\dot{g} \neq 0$. Hence, the solution to the equations for the brane configuration in this case is given by (3·38) and (3·40). The solution to the latter equation can be explicitly expressed in terms of a linear combination of $\sin(\sqrt{|\lambda|}T)$ and $\cos(\sqrt{|\lambda|}T)$ for $K\lambda < 0$ or $\sinh(\sqrt{|\lambda|}T)$ and $\cosh(\sqrt{|\lambda|}T)$ for $K\lambda > 0$, if necessary. Further, as described in Appendix C, X is uniquely determined, up to an isomorphism of M_K^{D-2} and multiplication by a constant, as a solution to

$$D_i D_j X = -KX\gamma_{ij}; \quad (DX)^2 = -KX^2 + c. \tag{3·42}$$

ii) The case $\lambda = 0$: In this case, K must be unity, which implies that $U = W = 1$, and r is related to R by

$$R = e^r. \tag{3·43}$$

Further, we can set $X = \cos\chi$ by a constant multiplication and an isometry, for which $c = 1$. Equation (3·37) can be easily integrated to give

$$X \equiv \cos\chi = -\frac{g}{R} \mp \frac{a\sigma}{2}R. \tag{3·44}$$

In this case, (3·33b) gives the condition

$$\mp 2a\sigma g = \dot{g}^2 + a^2 - 1, \tag{3·45}$$

and (3·33d) is written

$$\ddot{g} \pm a\sigma = 0. \tag{3·46}$$

Hence, the general solution to the brane configuration equations is given by

$$\sigma \neq 0 : R\cos\chi = \pm\frac{a\sigma}{2}\left[(T - T_0)^2 - R^2\right] + b;\ a^2 \mp 2ab\sigma = 1, \tag{3·47a}$$
$$\sigma = 0 : R\cos\chi = pT + A;\ p^2 < 1. \tag{3·47b}$$

(B) Solutions represented as $R = R(\boldsymbol{z})$

When $R_T \equiv 0$, R (and r) depends only on $X = X(\boldsymbol{z})$, and W is still given by $W = K - \lambda R^2$, but U need not coincide with W. Hence, the brane configuration equations are written

$$\frac{1}{r_X^2} = -2\alpha_0 X + KX^2 + a^2 - c, \tag{3·48a}$$
$$\frac{\alpha_0 - KX}{R} - \frac{Z}{r_X} = \pm a\sigma, \tag{3·48b}$$
$$\frac{V_r}{r_X} + (\alpha_0 - KX)V = 0. \tag{3·48c}$$

Since $r_{XX}/r_X^3 = \alpha_0 - KX$, the last equation can be easily integrated to give

$$V = \frac{k}{r_X}, \tag{3·49}$$

where k is an integration constant.

(B-1) The case $K = 0$

As in the corresponding case with $R_T \not\equiv 0$, we have $Z = \sqrt{-\lambda}$ ($\lambda < 0$) and $\sqrt{-\lambda} rR = -1$, while solutions for $\alpha_0 = 0$ and solutions for $\alpha_0 \neq 0$ are qualitatively different.

i) The case $\alpha_0 = 0$: In this case, (3·48b) is written $r_X = \mp\sqrt{-\lambda}/a\sigma$, which requires $\sigma < 0$ and can be easily integrated to yield

$$\frac{1}{R} = \mp\frac{\lambda}{a\sigma} X + A, \tag{3·50}$$

where A is a constant. Inserting this into (3·48a) gives

$$\lambda c = (\sigma^2 + \lambda) a^2. \tag{3·51}$$

Further, from (3·49), it follows that V is constant, which implies that we can set $U = W$ by a constant rescaling of T. We can see that this solution is the special case of (3·30) with $q = 0$, by choosing the representation $X = a\boldsymbol{p} \cdot \boldsymbol{z}$, for which $c = a^2 \boldsymbol{p}^2$.

ii) The case $\alpha_0 \neq 0$: In this case, as in the corresponding case with $R_T \neq 0$, we can set $c = 0$ and $\alpha_0 = 2$, for which $X = (\boldsymbol{z} - \boldsymbol{z}_0)^2$ up to an isometry of E^{D-2}. Equation (3·48b) can be integrated to yields

$$r^2 + X = \mp\frac{a\sigma}{\sqrt{-\lambda}} r + b, \tag{3·52}$$

where b is an integration constant. This constant is determined by (3·48b) as

$$b = \frac{\sigma^2 + \lambda}{4\lambda} a^2. \tag{3·53}$$

Hence, we obtain

$$X \equiv (\boldsymbol{z} - \boldsymbol{z}_0)^2 = \frac{1}{\lambda}\left(\frac{1}{R} \mp \frac{a\sigma}{2}\right)^2 + \frac{a^2}{4}. \ (\sigma \neq 0) \tag{3·54}$$

From this expression and (3·49), $U = R^2 V^2$ is determined up to a constant factor as

$$U = \left(1 \mp \frac{a\sigma}{2} R\right)^2. \tag{3·55}$$

Hence, U is not proportional to W.

(B-2) The case $K \neq 0$

As in the corresponding case with $R_T \not\equiv 0$, we can set $\alpha_0 = 0$. Then, from (3·48a), rewritten as

$$\left(\frac{KX}{R} + \frac{Z}{r_X}\right)^2 - K\left(ZX + \frac{1}{Rr_X}\right)^2 = -\lambda(a^2 - c), \tag{3·56}$$

and (3·48b), we obtain

$$ZX + \frac{1}{Rr_X} = b;\ b^2 = K[(\sigma^2 + \lambda)a^2 - \lambda c]. \tag{3·57}$$

i) The case $\lambda \neq 0$: In this case, (3·57) and (3·48b) can be regarded as linear equations for X and $1/r_X$. Solving these equations and using (3·49), we obtain

$$U = (bK \pm a\sigma\sqrt{W})^2, \tag{3·58a}$$

$$\lambda X = -b\frac{\sqrt{W}}{R} \mp \frac{a\sigma}{R}, \tag{3·58b}$$

$$b^2 = K[(\sigma^2 + \lambda)a^2 - \lambda c], \tag{3·58c}$$

after an appropriate rescaling of U. U is proportional to W only when $b = 0$, for which the solution is the special case of (3·38) with $g = 0$.
ii) The case $\lambda = 0$: In this case, $K = 1$, $W = 1$, and $R = e^r$. The integration of (3·48b) yields

$$X = \mp\frac{a\sigma}{2}R + \frac{b}{R}. \tag{3·59}$$

Also, (3·48a) is written

$$c = a^2 \mp 2ab\sigma, \tag{3·60}$$

and (3·49) determines U as

$$U = \left(b \pm \frac{a\sigma}{2}R^2\right)^2, \tag{3·61}$$

after a constant rescaling. Since $c > 0$ in this case, U is proportional to $W = 1$ only when $\sigma = 0$, for which the solution is the special case of (3·47b) with $p = 0$.

To summarize, a static configuration with $R_i \neq 0$ is allowed only in bulk geometries for which $W = K - \lambda R^2$ and $U = W$ or U is given by (3·55)($K = 0$), (3·58a)($b\lambda \neq 0$) or (3·61)($\lambda = 0, b \neq 0$).

3.2. $\chi = \chi(T, \boldsymbol{\theta})$*-type configurations*

In this case, the equations for the brane configuration are given by (2·14) with $\chi_R \equiv 0$. In particular, from (2·14c) it follows that only the brane with $\sigma = 0$ can have this type of configuration. Hence, the brane configuration equations reduce to the following set of equations:

$$\chi_{TT} = 0, \tag{3·62a}$$

$$V'\chi_T = 0, \tag{3·62b}$$

$$\chi_{TA} - \frac{\rho'}{\rho}\chi_T\chi_A = 0, \tag{3·62c}$$

$$\hat{D}_A\hat{D}_B\left(\frac{\rho'}{\rho}\right) = -\frac{\rho'}{\rho}\gamma_{AB}, \tag{3·62d}$$

$$N^2 = 1 - \frac{\chi_T^2}{V^2} + \frac{1}{\rho^2}(\hat{D}\chi)^2. \tag{3·62e}$$

In deriving (3·62d) from (2·14f), we have used the identity

$$\left(\frac{\rho'}{\rho}\right)' = -\frac{1}{\rho^2}, \tag{3·63}$$

which holds for any value of K.

3.2.1. Solutions with $\chi_T \equiv 0$

For the static case, the brane configuration is determined by (3·62d). As shown in Appendix C, by an appropriate $O(D-2)$ transformation, any solution to this equation can be written

$$\frac{\rho'}{\rho} = a\cos\theta, \tag{3·64}$$

where a is a constant and θ is the geodesic distance from the north pole of the $(D-3)$-dimensional unit sphere. Note that for $K=-1$, we have $a^2 > 1$, since $\cosh\chi/\sinh\chi > 1$. No constraint on U or W arises.

3.2.2. Solutions with $\chi_T \not\equiv 0$

In the non-static case, (3·62b) requires V' to vanish. Hence, after an appropriate rescaling of T, U is written

$$U = R^2. \tag{3·65}$$

Further, from the argument in Appendix B, (3·62c) implies that χ depends on θ^A only through some function $M(\theta^A)$, and hence ρ'/ρ is written $\rho'/\rho = F(T, M(\theta^A))$. Inserting this expression into (3·62d) and using an argument similar to that concerning the T-independence of α and β in (3·10), we find that F can be written $F = A(T)M(\theta^A)$ through an appropriate redefinition of $M(\theta^A)$. Hence, we obtain

$$\frac{\rho'}{\rho} = A(T)\cos\theta, \tag{3·66}$$

where θ is the same as in (3·64). Inserting this expression into (3·62c) and using the identity $K\rho^2 + (\rho')^2 = 1$, we find that $K = 0$. This implies that the bulk metric can be expressed as

$$ds_D^2 = \frac{dR^2}{W(R)} + R^2(-dT^2 + d\boldsymbol{z}^2). \tag{3·67}$$

Further, (3·62a) determines A to be $A = 1/(aT+b)$, where a and b are constant. Hence, the general solution is expressed as

$$aT + b = \chi\cos\theta;\ a^2 < 1, \tag{3·68}$$

where the condition on a comes from $N^2 = (1-a^2)/\cos^2\theta > 0$. Since $\chi\cos\theta$ can be expressed as $\boldsymbol{n}\cdot\boldsymbol{z}$ in terms of a unit vector $\boldsymbol{n}$ in E^{D-2}, this solution corresponds to a time-like hyperplane in $E^{D-2,1}$ with the coordinates $(T, \boldsymbol{z})$ on each $R=$const section.

§4. Solutions in spacetimes with $S = S_0$

In this section, we solve the brane configuration equations for the Nariai-type bulk geometry whose metric is given by (2·9) with $W = 1$ and $S = S_0$. The arguments run almost parallel to those in the previous section.

4.1. *$R = R(T, \boldsymbol{z})$-type configurations*

In this case, (2·13) reads

$$R_{TT} - \frac{U'}{U}R_T^2 + \frac{1}{2}U' = \sigma N(-U + R_T^2), \tag{4·1a}$$
$$R_{Ti} - \frac{U'}{2U}R_T R_i = \sigma N R_T R_i, \tag{4·1b}$$
$$D_i D_j R = \sigma N(R_i R_j + S_0^2 \gamma_{ij}), \tag{4·1c}$$
$$N^2 = 1 - \frac{R_T^2}{U} + \frac{R_i R_j \gamma^{ij}}{S_0^2}. \tag{4·1d}$$

4.1.1. Configurations represented as $R = R(T)$

We first look for special solutions with $R_i \equiv 0$. From (4·1c), we see that such a solution exists only for $\sigma = 0$, and that (4·1a) is the only non-trivial equation. For configurations with $R_T \equiv 0$, this equation reduces to the equation $U' = 0$ for constant values of R. On the other hand, for configurations with $R_i \not\equiv 0$, (4·1a) is equivalent to

$$\left(\frac{R_T^2}{U^2} - \frac{1}{U}\right)_T = 0. \tag{4·2}$$

Hence, $U(R)$ can be an arbitrary function, and the brane configuration is described by a solution to the first-order ordinary equation

$$R_T^2 = U(1 - AU);\ \ A > 0, \tag{4·3}$$

where the condition $A > 0$ comes from $N^2 = AU > 0$. Thus, the general solution contains one free parameter in addition to the one representing time translation.

4.1.2. Configurations with $R_i \not\equiv 0$

Next, in the case $R_i \not\equiv 0$, (4·1b) implies that R is written $R = R(T, X(\boldsymbol{z}))$ and gives

$$R_{TX} = \left(\frac{U'}{2U} + \sigma N\right) R_T R_X. \tag{4·4}$$

Hence, repeating the argument of the previous section, we find that (4·1c) is equivalent to the set of equations

$$D_i D_j X = (\alpha_0 - KX)\gamma_{ij}, \tag{4·5a}$$
$$(DX)^2 = -KX^2 + 2\alpha_0 X + c, \tag{4·5b}$$

$$S_0^2 \frac{\sigma N}{R_X} = \alpha_0 - KX, \tag{4·5c}$$

$$\frac{R_{XX}}{R_X^3} = \frac{\alpha_0 - KX}{S_0^2}. \tag{4·5d}$$

By using these equations, (4·4) can be integrated to yield

$$R_T = V R_X \dot{g}(T), \tag{4·6}$$

where

$$V = \sqrt{U}/S_0, \tag{4·7}$$

and $g(T)$ is an arbitrary function of T. Also, integrating (4·5d) once gives

$$-\frac{S_0^2}{R_X^2} = -KX^2 + 2\alpha_0 X + c - f^2 - \dot{g}^2, \tag{4·8}$$

where f is a function of T. It can be assumed to be positive definite, because N is expressed as

$$S_0 \frac{N}{R_X} = \pm f. \tag{4·9}$$

Inserting this expression into (4·5c), we obtain

$$\alpha_0 - KX = \pm S_0 \sigma f(T), \tag{4·10}$$

from which it follows that $K = 0$. Therefore, from this point, we set $S_0 = 1$ by redefining the Euclidean metric $S_0^2 \gamma_{ij}$ as γ_{ij}. Then, the bulk metric is given by

$$ds_D^2 = -U(R)dT^2 + dR^2 + d\boldsymbol{z}^2, \tag{4·11}$$

and the spatial section becomes the Euclidean space E^{D-1}.

With the help of these equations, (4·1a) can be deformed into

$$\ddot{g} + \frac{1}{R_X^3}(V R_X)_X = 0. \tag{4·12}$$

Also, differentiation of (4·8) with respect to T yields

$$-f\dot{f} = \dot{g}\left(\ddot{g} + \frac{1}{R_X^3}(V R_X)_X\right). \tag{4·13}$$

Hence, f must be a positive constant a, which is related to α_0 as

$$\alpha_0 = \pm a\sigma. \tag{4·14}$$

To summarize, the equations for the brane configuration are reduced to (4·6), (4·8) with $f = a$ and $K = 0$, (4·12), and (4·14).

As in the previous section, by considering $d(X+gV)$, we find that $X+gV$ depends only on R:

$$X+gV=h(R). \tag{4·15}$$

In order to determine $h(R)$ and $g(T)$, we regard X as a function of T and R. Then, since $X_R=1/R_X$, (4·8) is written

$$\dot{g}^2-(V')^2g^2+2(\alpha_0 V+h'V')g+a^2-c-2\alpha_0 h-(h')^2=0. \tag{4·16}$$

(A) Solutions with $R_T\not\equiv 0$ and $R_i\not\equiv 0$

Let us first consider the case $\dot{g}\not\equiv 0$. In this case, since T and R are independent variables, (4·16) leads to

$$V'=\text{const},\ \ \alpha_0 V+h'V'=\text{const}, \tag{4·17}$$

$$-a^2+c+2\alpha_0 h+(h')^2=b, \tag{4·18}$$

where b is a constant. Solutions to these equations differ qualitatively depending on whether $V'\equiv 0$ or $V'\not\equiv 0$.

(A-1) The case $U=1$

In the case $V'\equiv 0$, we can set $U=1$ by rescaling T. Hence, the bulk metric (4·11) becomes the standard Minkowski metric. Since (4·18) implies $h''+\alpha_0=0$, its general solution is easily found to be

$$h=-\frac{1}{2}\alpha_0 R^2+h_1 R+h_0;\quad b=c-a^2+2\alpha_0 h_0+h_1^2, \tag{4·19}$$

where h_0 and h_1 are integration constants. Further, (4·16) reads

$$\dot{g}^2+2\alpha_0 g=b. \tag{4·20}$$

Since $\dot{g}\not\equiv 0$, (4·12) follows from this equation.

For $\sigma=0$, we have $\alpha_0=0$ from (4·14), and X is written $X=\boldsymbol{p}\cdot\boldsymbol{z}$ with $\boldsymbol{p}^2=c$ for a Cartesian coordinate system $\boldsymbol{z}$. Further, the general solution to (4·20) is $\pm\sqrt{b}T+\text{const}$. Hence, from (4·15) and (4·19), the general solution to the brane configuration equations is given by

$$sT=\boldsymbol{p}\cdot\boldsymbol{z}+qR+A;\ s^2<\boldsymbol{p}^2+q^2, \tag{4·21}$$

where s,p_i,q and A are constants. This solution represents a time-like hyperplane in the Minkowski spacetime $E^{D-1,1}$.

On the other hand, for $\sigma\neq 0$, we can set $X=(\boldsymbol{z}-\boldsymbol{z}_0)^2$, for which $\alpha_0=2$ and $c=0$. The general solution to (4·20) now reads $g=-(T-T_0)^2-a^2/4+h_0+h_1^2/4$. Hence, the solution to the brane configuration equations is given by

$$(\boldsymbol{z}-\boldsymbol{z}_0)^2+(R-R_0)^2=(T-T_0)^2+\frac{1}{\sigma^2}, \tag{4·22}$$

which represents a time-like hyperboloid in the Minkowski spacetime $E^{D-1,1}$.

(A-2) The case $U = R^2$

Next, we consider the case $V' \not\equiv 0$. In this case, we can set $U = R^2$ by rescaling T. Hence, the bulk metric (4·11) coincides with the Minkowskian metric in the Rindler coordinates:

$$ds_D^2 = -R^2 dT^2 + dR^2 + d\boldsymbol{z} \cdot d\boldsymbol{z} = -(d(R\sinh T))^2 + (d(R\cosh T))^2 + d\boldsymbol{z} \cdot d\boldsymbol{z}. \tag{4·23}$$

First, we consider a brane with $\sigma = 0$. In this case, $\alpha_0 = 0$ from (4·14), and $h = h_1 R + h_0$ from (4·17). We can set $h_1 = 0$ by redefining $g - h_1$ as g, from (4·15). Hence, X is written

$$X = h_0 - gR, \tag{4·24}$$

where g is a solution to

$$\dot{g}^2 - g^2 + a^2 - c = 0, \tag{4·25}$$

from (4·20). Thus, the general solution to the brane configuration equations is given by

$$\boldsymbol{p} \cdot \boldsymbol{z} = qR\cosh T + sR\sinh T + A; \; \boldsymbol{p}^2 + q^2 > s^2. \tag{4·26}$$

From (4·23), we see that this solution represents a time-like hyperplane in the Minkowski spacetime.

Next, for $\sigma \neq 0$, by setting $\alpha_0 = 2$ and $c = 0$, we find that h is expressed as $h = -R^2 + h_0$ after a constant shift of g. The value of g is determined from

$$\dot{g}^2 - g^2 + a^2 - 4h_0 = 0. \tag{4·27}$$

Hence, the general solution to the equations for the brane configuration is expressed as

$$\boldsymbol{z} \cdot \boldsymbol{z} + (R\cosh T - A)^2 = (R\sinh T - B)^2 + \frac{1}{\sigma^2}. \tag{4·28}$$

This is a time-like hyperboloid in the Minkowski spacetime, represented in the Rindler coordinates.

(B) Solutions represented as $R = R(\boldsymbol{z})$

Finally, we consider the static case, $\dot{g} \equiv 0$. In this case, we can set $g \equiv 0$ without loss of generality. Then, $X = h(R)$ and U is proportional to $X_R^2 = (h')^2$ from (4·12). The solution for h is now given by (4·19) with $b = 0$ from (4·16). It follows from these that solutions to the brane configuration equations in the present case are simply given by setting $s = 0$ in (4·21) for $\sigma = 0$ and $A = B = 0$ in (4·28) for $\sigma \neq 0$.

To summarize, a bulk spacetime with $W = 1$ and $S = S_0$ can contain a vacuum brane with a configuration $R_i \not\equiv 0$ only when it is a Minkowski spacetime, and allowed configurations of the brane are hyperplanes for $\sigma = 0$ and hyperboloids for $\sigma \neq 0$.

4.2. $\chi = \chi(T, \boldsymbol{\theta})$*-type configurations*

For $S = S_0$ and $\chi_R \equiv 0$, it is easy to see that (2·14c) leads to $\sigma = 0$ and the rest of Eqs. (2·14) coincide with the corresponding equations (3·62) for $S = R$ if we replace V by U. Hence, solutions to the brane configuration equations of the type $\chi = \chi(T, \boldsymbol{\theta})$ are given by two families defined by

$$U \text{ is arbitrary},\ K = 0, \pm 1,\ \frac{\rho'}{\rho} = a\cos\theta, \tag{4·29a}$$

and

$$U = 1,\ K = 0,\ aT + b = \chi\cos\theta;\ a^2 < 1, \tag{4·29b}$$

up to transformations by isometries of the bulk spacetime. Note thatthe former family represents a totally geodesic hypersurface in theconstant curvature space M_K^{D-2}, as in the case with $S = R$, whilethe latter family is a special case of (4·21) with $q = 0$.

§5. Classification of the bulk geometry and the brane configuration

In this section, we classify the solutions obtained in the previous two sections by identifying solutions connected by isometries. Throughout this section, the relation $M^D = (B, F)$ signifies that the bulk geometry of M^D is a warped product of a base space B and a fibre F, i.e.,

$$ds_D^2 = g_{ab}(y)dy^a dy^b + f(y)^2\gamma_{ij}(z)dz^i dz^j. \tag{5·1}$$

We first group the solutions obtained in §§3 and 4 into the following four types, mainly according to the symmetry property of the bulk geometry.

Type I: This type is defined as the set of solutions for which the brane configuration is expressed in one of the special forms (A), (B) and (C) described below, in some coordinate system in which the bulk metric is written in the form (2·9). Except for the $IO(1) \times G(D-2, K)$ symmetry, no additional symmetry is assumed on the bulk geometry, and hence the bulk spacetime has the structure $M^D = (N^2, M_K^{D-2})$, where N^2 is the static 2-dimensional spacetime.

(A) Static configurations of a brane with $\sigma = 0$ expressed as

$$\frac{\rho'}{\rho} = a\cos\theta. \tag{5·2}$$

(B) Configurations with $\sigma \neq 0$ of the form $R = R(T)$ satisfying

$$R_T^2 = U^2\left(1 - \frac{U}{\sigma^2 R^2}\right) \not\equiv 0 \tag{5·3}$$

in special bulk geometries with $S = R$ and $U = W$, and those of a brane with $\sigma = 0$ of the form $R(T)$ satisfying

$$R_T^2 = WU(1 - AU) \not\equiv 0;\ A > 0 \tag{5·4}$$

in bulk geometries with S =const.

(C) Static brane configurations expressed as R =const in terms of a solution to

$$\frac{U'}{U} = \frac{2S'}{S}, \quad W\left(\frac{S'}{S}\right)^2 = \sigma^2. \tag{5·5}$$

Type II: This type is defined by the condition that the bulk geometry is of the type $M^D = (M_\lambda^{D-1}, E^{0,1})$ and the metric is written

$$ds_D^2 = d\sigma_{\lambda,D-1}^2 - U(R)dT^2, \tag{5·6}$$

where $d\sigma_{\lambda,D-1}^2$ is the metric of a $(D-1)$-dimensional constant curvature space M_λ^{D-1} with sectional curvature λ. R is a coordinate in this space such that each R =const surface has a constant curvature with the same sign as that of K, and the metric is written in the form (2·9). When $U(R)$ is proportional to $W(R) = K - \lambda R^2$ for $S = R$ or $U = 1$ for $\lambda = 0$, the spacetime becomes a constant curvature spacetime with sectional curvature λ. This case is excluded, in order to make this type mutually exclusive with the type IV. We also exclude bulk geometries for which $d\sigma_{\lambda,D-1}^2 = dR^2 + d\boldsymbol{z}^2$, in order to make this type mutually exclusive with the type I-B.
Type III: This type is defined by the condition that the bulk geometry has the symmetry $IO(D-2,1)$ and $M^D = (E^1, E^{D-2,1})$. The metric is written

$$ds_D^2 = dv^2 + f(v)^2\eta_{\mu\nu}dx^\mu dx^\nu. \tag{5·7}$$

When $f(v)$ is proportional to e^{av}, the geometry becomes $E^{D-1,1}$ for $a = 0$ and AdS^D for $a \neq 0$. These cases are excluded in order to make this type mutually exclusive with the type IV.
Type IV: This type is defined by the condition that the bulk spacetime is a constant curvature spacetime, i.e., $M^D = E^{D-1,1}$, dS^D or AdS^D. In the representation (2·9), this type corresponds to the cases in which $U = W = K - \lambda R^2$ and $S = R$, where λ coincides with the sectional curvature of the bulk from (A·4), or $K = 0$, $U = 1$ or R^2, $W = 1$ and S =const.

The solutions for $R_T \not\equiv 0$ and $R_i \not\equiv 0$ with $S = R$ in §3, and those for $R_i \not\equiv 0$ or $\chi_T \not\equiv 0$ with $S = 1$ in §4 belong to the type IV. The solutions with $\chi_T \not\equiv 0$ in §3 belong to the type III. As shown there, these solutions correspond to brane configurations with $\sigma = 0$ that have the structure $\Sigma = (E^1, E^{D-3,1})$, where $E^{D-3,1}$ is an arbitrary time-like hyperplane of $E^{D-2,1}$. The solutions with $R_T \equiv 0$ and $R_i \not\equiv 0$ in §3 belong to the type II. The solutions with $\chi_T \equiv 0$ and $\chi_R \equiv 0$ in §§3 and 4 belong to one of the types I-A, II, III or IV, depending on the bulk geometry. The solutions with $R_i = 0$ in §§3 and 4 belong to either the type I-B or the type I-C, depending on whether $R_T \not\equiv 0$ or $R_T \equiv 0$.

This correspondence can be rephrased in the following way from the point of view of the symmetry of a brane. Let $IO(1) \times G_0$ be a subgroup of $\mathrm{Isom}(M^D)$ such that G_0 is isomorphic to $G(D-2, K)$. Then, first, if a brane is $IO(1) \times G_0$ invariant, solutions are of the type I-C, and σ is restricted to some special values for analytic metrics. Second, if a brane is G_0 invariant but not $IO(1)$ invariant, solutions are of the type I-B, and the bulk geometry is constrained. Third, if a brane is $IO(1)$

invariant but not G_0 invariant, solutions are of the type I-A, the type III with $\sigma = 0$, the type II or the type IV. In the latter three cases, the bulk geometry is quite special. Finally, if a brane is neither G_0 invariant nor $IO(1)$ invariant, solutions are of the type III with $\sigma = 0$ or of the type IV.

Here, note that the type I and the other types may not be mutually exclusive, although the types II, III and IV as well as the types I-B, II and III are mutually exclusive, as will be shown later. Actually, we show below that the types III, IV and most geometries of the type II are subclasses of the type I. Further, solutions belonging to different subtypes of the type I may be geometrically identical in the case in which the bulk spacetime allows two different space-time decomposition of the form (2·9).

The main purpose of the present section is to make these points clear. For this purpose, we use a general geometrical argument concerning everywhere umbilical hypersurfaces of a constant curvature space(-time), instead of starting from explicit expressions for the solutions in special coordinate systems obtained in the previous two sections.

5.1. *Everywhere umbilical hypersurfaces of a constant curvature space(-time)*

Let $\tilde{M}_\lambda$ be a constant curvature space(-time) with sectional curvature λ and Σ be an everywhere umbilical hypersurface, whose extrinsic curvature K_{ij} is proportional to the induced metric g_{ij}, $K_{ij} = \sigma g_{ij}$. In general, this condition is weaker than that imposed on a vacuum brane, because σ of an everywhere umbilical hypersurface need not be a constant. However, in a constant curvature space(-time), these two conditions become equivalent. This is an immediate consequence of the contracted Codazzi equation,

$$0 = n^\mu \tilde{R}_{\mu j i}{}^j = \mp(n-1)\nabla_i \sigma, \tag{5·8}$$

where n^μ is the unit vector normal to Σ with norm ± 1, and n is the dimension of $\tilde{M}_\lambda$. Hence, we can assume that σ is constant. Further, from the Gauss equation

$$R_{ijkl} = \tilde{R}_{ijkl} + K_{ik}K_{jl} - K_{il}K_{jk} = (\lambda + \sigma^2)(g_{ik}g_{jl} - g_{il}g_{jk}), \tag{5·9}$$

it follows that Σ has constant sectional curvature $\lambda + \sigma^2$. Hence, when λ is given, σ^2 uniquely determines the intrinsic geometry of Σ. In this section, we show that σ^2 also uniquely determines the configuration of Σ up to an isometric transformation of $\tilde{M}$, and thus an everywhere umbilical hypersurface in a constant curvature space(-time) is rigid. In this section, we assume that the hypersurface is time-like when $\tilde{M}_\lambda$ is a spacetime.

Let us consider a Gaussian normal coordinate system (v, x^i) in terms of which the metric is written

$$d\tilde{s}^2 = dv^2 + g_{ij}(v, \boldsymbol{x})dx^i dx^j, \tag{5·10}$$

and Σ is expressed as $v = v_0$ (constant). Then, the definition of K_{ij} and the decomposition of the curvature $\tilde{R}_{vivj}$ of $\tilde{M}$ yield

$$\partial_v g_{ij} = 2K_{ij}, \tag{5·11a}$$

$$\partial_v K_{ij} - K_{il}K^l_j = -\tilde{R}_{vivj} = -\lambda g_{ij}. \tag{5·11b}$$

With the ansatz $K_{ij} = k(v)g_{ij}$, these equations reduce to

$$\partial_v k + k^2 = -\lambda, \tag{5·12a}$$
$$\partial_v g_{ij} = 2k g_{ij}. \tag{5·12b}$$

The general solution to the first equation is given by

$$\lambda = 0: \quad k = \frac{1}{v},\ 0, \tag{5·13a}$$
$$\lambda = \frac{1}{\ell^2}: \ k = \frac{1}{\ell}\cot\frac{v}{\ell}, \tag{5·13b}$$
$$\lambda = -\frac{1}{\ell^2}: k = \frac{1}{\ell}\coth\frac{v}{\ell},\ \frac{1}{\ell},\ \frac{1}{\ell}\tanh\frac{v}{\ell}. \tag{5·13c}$$

For each $k(v)$, the second equation determines g_{ij} to be

$$g_{ij} = \ell^2\rho(v/\ell)^2\hat{g}_{ij}(\boldsymbol{x}), \tag{5·14}$$

where

$$\lambda = 0: \quad \rho(\chi) = \chi,\ 1, \tag{5·15a}$$
$$\lambda = \frac{1}{\ell^2}: \ \rho(\chi) = \sin\chi, \tag{5·15b}$$
$$\lambda = -\frac{1}{\ell^2}: \rho(\chi) = \sinh\chi,\ e^\chi,\ \cosh\chi. \tag{5·15c}$$

Since (5·11) is a set of ordinary differential equations with respect to the independent variable v for a fixed value of x^i, its solution is uniquely determined by $K_{ij}(v_0, \boldsymbol{x})$ and $g_{ij}(v_0, \boldsymbol{x})$. However, because they are related by $K_{ij} = \sigma g_{ij}$ at $v = v_0$, and because k given above can be always made to coincide with a given σ by choosing v_0 appropriately, (5·14) obtained with the ansatz $K_{ij} = k(v)g_{ij}$ actually yields the unique global solution to (5·11). Further, the Gauss equation guarantees that $\hat{g}_{ij}$ represents a normalized constant curvature metric on Σ:

$$\hat{R}_{ijkl} = K(\hat{g}_{ik}\hat{g}_{jl} - \hat{g}_{il}\hat{g}_{jk}); \tag{5·16}$$
$$K \equiv \ell^2\rho(v/\ell)^2(\lambda + k^2) = \begin{cases} 1, \quad 0 & : \lambda = 0, \\ 1 & : \lambda = 1/\ell^2, \\ 1, \quad 0, \quad -1 & : \lambda = -1/\ell^2. \end{cases} \tag{5·17}$$

Hence, the metric can be written

$$d\tilde{s}^2 = dv^2 + \ell^2\rho(v/\ell)^2 d\sigma_K^2. \tag{5·18}$$

If $\rho(\chi) \not\equiv 1$, by introducing the coordinate r as $r = \ell\rho(v/\ell)$, this metric can be put into the form

$$d\tilde{s}^2 = \frac{dr^2}{K - \lambda r^2} + r^2 d\sigma_K^2, \tag{5·19}$$

for which the location of the brane, $r = r_0$, and the brane tension σ are related by

$$\frac{K}{r_0^2} = \sigma^2 + \lambda. \tag{5·20}$$

Also, for $\rho(\chi) \equiv 1$, i.e., for $\lambda = K = 0$ and $\sigma = 0$, the metric can be written

$$d\tilde{s}^2 = dv^2 + \eta_{ij} dx^i dx^j, \tag{5·21}$$

where η_{ij} is a Kronecker delta or a Minkowski metric.

Note that the coordinate system in which the metric of $\tilde{M}_\lambda$ is written in the form (5·18) and Σ is represented as $v = v_0$ is unique if σ^2 and λ are given, except for the freedom corresponding to $v/\ell \to \pi - v/\ell$ for $\lambda > 0$ and the freedom corresponding to transformations of the form $v \to v+$const and an associated rescaling of the flat metric for $K = 0$. This implies that two locally isotropic hypersurfaces with the same value of σ^2 are mapped into each other by an isometry of $\tilde{M}$.

5.2. *Type IV solutions*

From the general argument in the previous subsection, it is clear that for the type IV solutions, for which the bulk spacetime is a constant curvature spacetime M_λ^D, a brane Σ^{D-1} with $K_{\mu\nu} = \sigma g_{\mu\nu}$ is a $(D-1)$-dimensional constant curvature spacetime M_K^{D-1} with $K = K_\Sigma \equiv \lambda + \sigma^2$, and its configuration is uniquely specified by the value of σ^2 up to isometries of the bulk spacetime. In this section, we give a general expression for this brane configuration in a coordinate system in which isometry transformations have the simplest representation. Then, using this general expression, we show that the type IV is actually a subclass of the type I.

5.2.1. $M^D = E^{D-1,1}$

For $\lambda = 0$, the bulk spacetime is $E^{D-1,1}$, and its metric is given by

$$E^{D-1,1}: \ ds_D^2 = -dT^2 + d\boldsymbol{x}^2. \tag{5·22}$$

The isometry group is represented by the standard Poincare group $IO(D-1,1)$ in this coordinate system (T, x^a). Since $K_\Sigma = \sigma^2 \geq 0$, this spacetime can contain two types of branes.

First, for $\sigma = 0$, $K_\Sigma = 0$ and the brane Σ^{D-1} is isometric to $E^{D-2,1}$. Its configuration is given by

$$\Sigma^{D-1} \cong E^{D-2,1} : qT + \boldsymbol{p}\cdot\boldsymbol{x} = A; \ \boldsymbol{p}^2 > q^2, \tag{5·23}$$

which can always be put into the form $x^{D-1} = 0$ by an isometry. If we set $R = x^{D-1}$ and introduce the spherical coordinates (χ, Ω_{D-3}^i) for $(x^1, \cdots, x^{D-2})$, the metric (5·22) can be written in the form (2·9) with $U = W = S = 1$ and $K = 1$. In this coordinate system, the brane configuration $x^{D-1} = A$ is written $R = A$ and belongs to the type I-C. On the other hand, if we transform the brane configuration to $x^{D-2} = A$ by an isometry, it can be written $\chi \cos\theta = A$, which belongs to the type I-A. Further, if we transform the configuration to $qT + |\boldsymbol{p}| X^{D-1} = A$, it belongs to the type I-B. Thus, every brane configuration with $\sigma = 0$ can be put into a form

belonging to the type I, and all solutions with $\sigma = 0$ of the type I are mapped into each other by isometries.

Next, for $\sigma \neq 0$, the curvature K_Σ of the brane is always positive and Σ^{D-1} is isometric to $\mathrm{dS}^{D-1}(1/|\sigma|)$. Its configuration is given by

$$\Sigma^{D-1} \cong \mathrm{dS}^{D-1}(1/|\sigma|):\ -(T-T_0)^2 + (\boldsymbol{x}-\boldsymbol{a})^2 = 1/\sigma^2. \tag{5·24}$$

In the spherical coordinates (R, Ω^i_{D-2}) for $\boldsymbol{x}$, the metric (5·22) takes the form (2·9) with $U = W = 1$, $S = R$ and $K = 1$. In this coordinate system, the configuration (5·24) with $\boldsymbol{a} \neq 0$ can be written in the form (3·47) by choosing the coordinate χ as $\boldsymbol{x} \cdot \boldsymbol{a} = R|\boldsymbol{a}|\cos\chi$, while it belongs to the type I-B for $\boldsymbol{a} = 0$. Hence, (5·24) is also isometric to a solution of the type I.

5.2.2. $M^D = \mathrm{dS}^D$

For $\lambda = 1/\ell^2 > 0$, the bulk spacetime is a de Sitter spacetime $\mathrm{dS}^D(\ell)$, and its metric is expressed in terms of a homogeneous coordinate system Y as

$$\mathrm{dS}^D:\ ds^2_D = \ell^2 dY\cdot dY;\ Y\cdot Y \equiv -(Y^0)^2 + (Y^1)^2 + \cdots + (Y^D)^2 = 1. \tag{5·25}$$

The isometry group is represented as the linear transformation group $O(D,1)$ in the Y-space. Since $K_\Sigma = \sigma^2 + 1/\ell^2 > 0$, Σ^{D-1} is isometric to a de Sitter spacetime $\mathrm{dS}^{D-1}(\ell/\sqrt{1+\ell^2\sigma^2})$. If we define a new coordinate system $(r, \bar{Y}^M)$ by

$$r = \ell\sqrt{1-(Y^D)^2},\ \bar{Y}^M = \frac{Y^M}{\sqrt{1-(Y^D)^2}},\ (M = 0, \cdots, D-1) \tag{5·26}$$

the metric (5·25) takes the form of (5·19), with $d\sigma^2_K$ given by

$$ds^2_{D-1} = d\bar{Y}\cdot d\bar{Y};\ \bar{Y}\cdot\bar{Y} \equiv -(\bar{Y}^0)^2 + (\bar{Y}^1)^2 + \cdots + (\bar{Y}^{D-1})^2 = 1. \tag{5·27}$$

The brane configuration is isometric to the hypersurface $r = \ell/\sqrt{1+\ell^2\sigma^2}$ in this coordinate system, and to $Y^D = \pm\ell\sigma/\sqrt{1+\ell^2\sigma^2}$ in the Y-coordinate system. Hence, the general configuration of a brane is represented as

$$\Sigma^{D-1} \cong \mathrm{dS}^{D-1}\left(\frac{\ell}{\sqrt{1+\ell^2\sigma^2}}\right):\ P\cdot Y = \frac{\ell\sigma}{\sqrt{1+\ell^2\sigma^2}};\ P\cdot P = 1. \tag{5·28}$$

The metric (5·25) can be put into the form (2·9) with $U = W = 1 - R^2/\ell^2$, $S = R$ and $K = 1$ in the coordinates (T, R, Ω^a_{D-2}) defined by

$$Y^0 = \sqrt{1-\frac{R^2}{\ell^2}}\sinh\frac{T}{\ell},\ Y^D = \sqrt{1-\frac{R^2}{\ell^2}}\cosh\frac{T}{\ell},\ Y^a = \frac{R}{\ell}\Omega^a.\ (a = 1, \cdots, D-1) \tag{5·29}$$

Since $X = p_a\Omega^a$ for any constant vector p_a satisfies (3·42) with $K = 1$ and $c = p^a p_a$, as shown in Appendix C, it is easy to see that (5·28) can be written in the form (3·38) when P^0 or P^D is not zero and $(P^a) \neq 0$, while it takes the type I-B form when $P^a = 0$. Further, it belongs to the type I-A when $\sigma = 0$ and $P^0 = P^D = 0$. There exists no representation of the type I-C.

5.2.3. $M^D = \mathrm{AdS}^D$

Finally, for $\lambda = -1/\ell^2 < 0$, the bulk spacetime is an anti-de Sitter spacetime $\mathrm{AdS}^D(\ell)$, and its metric is expressed in terms of a homogeneous coordinate system Z as

$$\mathrm{AdS}^D: \ ds_D^2 = \ell^2 dZ \cdot dZ; \ Z \cdot Z \equiv -(Z^0)^2 - (Z^D)^2 + (Z^1)^2 + \cdots + (Z^{D-1})^2 = -1. \tag{5·30}$$

The isometry group is represented as the linear transformation group $O(D-1,2)$ in the Z-space. In this case, the curvature $K_\Sigma = \sigma^2 - 1/\ell^2$ of the brane can take any value.

First, for $\sigma^2\ell^2 = 1$, $K_\Sigma = 0$ and the brane is isometric to $E^{D-2,1}$. In the coordinates (r, x^a) defined by

$$r = \ell|Z^D - Z^{D-1}|, \ x^a = \frac{Z^a}{Z^D - Z^{D-1}}, \ (a = 0, \cdots, D-2) \tag{5·31}$$

(5·30) takes the form of (5·19) with $d\sigma_K^2 = d\boldsymbol{x} \cdot d\boldsymbol{x}$. Hence, the general configuration of a brane is expressed as

$$\Sigma^{D-1} \cong E^{D-2,1} : P \cdot Z = A; \ P \cdot P = 0, \tag{5·32}$$

where A is an arbitrary non-vanishing constant, which can be set to unity by a boost in the $Z^D - Z^{D-1}$ plane. If we introduce the coordinates (T, R, z^i) by

$$R = r, \ T - T_0 = \frac{\ell^2 Z^0}{r}, \ z^i = \frac{\ell Z^i}{r} \ (i = 1, \cdots, D-2), \tag{5·33}$$

the metric (5·30) takes the form of (2·9) with $U = W = R^2/\ell^2$, $S = R$ and $K = 0$. In this coordinate system, the configuration (5·32) gives the solution (3·32) if $P^D \neq P^{D-1}$, while it gives the solution (3·30) if $P^D = P^{D-1}$ and $(P^\mu) \neq 0$ ($\mu = 0, \cdots, D-2$). Also, for $P^D = P^{D-1}$ and $P^\mu = 0$, the brane configuration becomes $R =$const and belongs to the type I-C. There exists no representation of the type I-B.

Second, for $\sigma^2\ell^2 > 1$, $K_\Sigma > 0$ and the brane is isometric to $\mathrm{dS}^{D-1}(\ell/\sqrt{\ell^2\sigma^2 - 1})$. In this case, in the coordinate system $(r, \bar{Y}^M)$ defined by

$$r = \ell\sqrt{(Z^D)^2 - 1}, \ \bar{Y}^M = \frac{Z^M}{\sqrt{(Z^D)^2 - 1}}, \ (M = 0, \cdots, D-1) \tag{5·34}$$

the metric (5·30) takes the form of (5·19) with $d\sigma_K^2$ given by (5·27). Hence, the brane configuration is

$$\Sigma^{D-1} \cong \mathrm{dS}^{D-1}\left(\frac{\ell}{\sqrt{\ell^2\sigma^2 - 1}}\right): \ P \cdot Z = \frac{\ell\sigma}{\sqrt{\ell^2\sigma^2 - 1}}; \ P \cdot P = -1. \tag{5·35}$$

In the coordinate system (T, R, Ω^a) defined by

$$Z^0 = \sqrt{1 + \frac{R^2}{\ell^2}} \sin\frac{T}{\ell}, \ Z^D = \sqrt{1 + \frac{R^2}{\ell^2}} \cos\frac{T}{\ell}, \ Z^a = \frac{R}{\ell}\Omega^a, \ (a = 1, \cdots, D-1) \tag{5·36}$$

Table I. Type I configurations in the type IV geometry.

Spacetime	σ	Type I configurations
$E^{D-1,1}$:	$\sigma=0$	I-A, I-B, I-C (mutually isometric)
	$\sigma\neq 0$	I-B
dS^D:	$\sigma=0$	I-A, I-B (mutually isometric)
	$\sigma\neq 0$	I-B
AdS^D:	$\sigma=0$	I-A
	$\sigma\neq 0$	I-B ($\ell^2\sigma^2\neq 1$) I-C ($\ell^2\sigma^2=1$)

the metric (5·30) can be written in the form of (2·9) with $U=W=1+R^2/\ell^2$, $S=R$ and $K=1$. When (P^a) does not vanish, the brane configuration (5·35) in this coordinate system gives (3·38) by setting $X=P_a\Omega^a$, while for $(P^a)=0$, it gives the type I-B solution. There exists no representation of the type I-C.

Finally, for $\sigma^2\ell^2<1$, $K_\Sigma<0$ and the brane is isometric to $\mathrm{AdS}^{D-1}(\ell/\sqrt{1-\ell^2\sigma^2})$. In the coordinate system $(r,\bar{Z}^M)$ defined by

$$r=\ell\sqrt{(Z^{D-1})^2+1},\ \bar{Z}^M=\frac{Z^M}{\sqrt{(Z^{D-1})^2+1}},\ (M=0,\cdots,D-2,D) \tag{5·37}$$

the metric (5·30) takes the form of (5·19) with $d\sigma_K^2$ given by

$$ds^2_{D-1}=d\bar{Z}\cdot d\bar{Z};\ \bar{Z}\cdot\bar{Z}\equiv-(\bar{Z}^0)^2-(\bar{Z}^D)^2+(\bar{Z}^1)^2+\cdots(\bar{Z}^{D-2})^2=-1. \tag{5·38}$$

Hence, the brane configuration is given by

$$\Sigma^{D-1}\cong\mathrm{AdS}^{D-1}\left(\frac{\ell}{\sqrt{1-\ell^2\sigma^2}}\right):\ P\cdot Z=\frac{\ell\sigma}{\sqrt{1-\ell^2\sigma^2}};\ P\cdot P=1. \tag{5·39}$$

In the coordinate system $(T,R,\bar{Y}^a)$ defined by

$$\begin{aligned}Z^0&=\sqrt{-1+\frac{R^2}{\ell^2}}\sinh\frac{T}{\ell},\ Z^{D-1}=\sqrt{-1+\frac{R^2}{\ell^2}}\cosh\frac{T}{\ell},\\ Z^a&=\frac{R}{\ell}\bar{Y}^a,\ (a=1,\cdots,D-2,D)\end{aligned} \tag{5·40}$$

the metric (5·30) takes the form of (2·9) with $U=W=-1+R^2/\ell^2$, $S=R$ and $K=-1$. Since $X=p_a\bar{Y}^a$ for any constant vector p_a satisfies (3·42) with $K=-1$ and $c=p^ap_a$, as shown in Appendix C, the brane configuration (5·35) in this coordinate system gives (3·38) when (P^a) does not vanish, while for $(P^a)=0$ and $\sigma\neq 0$ it gives the type I-B solution. Further, for $P^0=P^{D-1}=0$ and $\sigma=0$, it can be written in the type I-A form. There exists no representation of the type I-C for $\sigma\neq 0$ and of the type I-B or the type I-C for $\sigma=0$.

5.2.4. Summary of the type IV solutions

The results obtained in this subsection can be summarized as follows. First, for a brane with $\sigma = 0$, solutions are mutually isometric and always have representations of the type I-A. They also have type I-B representations, except for $M^D = \mathrm{AdS}^D$, but a type I-C representation exists only for $M^D = E^{D-1,1}$. Second, for a brane with $\sigma \neq 0$, solutions with the same σ^2 are mutually isometric and have only representations of the type I-B, except for the brane with $\ell^2\sigma^2 = 1$ in AdS^D. In this exceptional case, there exist only type I-C representations. In order to make the comparison of these features with those for the other types easier, we list the possible type I representations for each constant curvature spacetime in Table I.

Note also that every brane configuration is invariant under some group isomorphic to $IO(D-2,1)$, $O(D-1,1)$ or $O(D-2,2)$. In particular, they are static, but not invariant under $IO(1) \times G(D-2,K)$, except in the $IO(D-2,1)$-invariant case.

5.3. *Type III solutions*

The type III geometry can be written

$$ds_D^2 = -R^2 dT^2 + \frac{dR^2}{W} + R^2 d\boldsymbol{z}^2, \tag{5·41}$$

where W is not proportional to R^2. Then, from the arguments in §3, the following four families of solutions can exist. The first family represents a brane with $\sigma = 0$ and configuration $R =$const. This is of the type I-C and exists only when $f'(v) = 0$ has a solution. The second family represents a brane with configuration $R = R(\boldsymbol{z})$ with $R_i \not\equiv 0$, which belongs to the type II. Since the type II and the type III are mutually exclusive, as shown below, this case cannot be realized. The third family represents a brane with $\sigma = 0$ belonging to the type I-A, which exists for any bulk geometry. The final family is a solution with $\sigma = 0$ and configuration $\chi = \chi(T, \boldsymbol{z})$ with $\chi_T \not\equiv 0$. It must correspond to a time-like hyperplane in the x-space $E^{D-2,1}$, and can always be expressed in the form I-A corresponding to the third family, in an appropriate coordinate system.

We thus find that a bulk spacetime of the type III allows only branes with $\sigma = 0$. Branes can have two geometrically different types of configurations. The first one is of the type I-A, which always exists. Configurations of this type are mutually isometric and do not have a I-B type representation, as shown below. The second one is of the type I-C, which is allowed only when $f'(v) = 0$ has a solution. Solutions of this type are not isometric to those of the type I-A.

5.4. *Type II solutions*

The bulk geometry of this type is described by (5·6) and has the structure $(E^{D-1}, E^{0,1})$, $(S^{D-1}, E^{0,1})$ or $(H^{D-1}, E^{0,1})$, depending on the sign of λ. Since this type is mutually exclusive with the types III and IV, as shown below, from the arguments at the beginning of this section, a solution belonging to this type must have a type I-B representation if it is not static. However, the bulk geometry is of the type IV if $U = W$. Further, only $IO(D-2)$-invariant branes with $\sigma = 0$ in $(E^{D-1}, E^{0,1})$ can have non-static configurations of the type I-B, but the bulk

Table II. Type I configurations in the type II geometry.

Spacetime	σ	Type I configurations
$(E^{D-1}, E^{0,1})$:	$\sigma = 0$	I-A
	$\sigma \neq 0$	I-C ($U \not\propto ((\boldsymbol{x}-\boldsymbol{a})^2+k)^2$)
$(S^{D-1}, E^{0,1})$:	$\sigma = 0$	I-A (& I-C)
	$\sigma \neq 0$	I-C ($U \not\propto (P\cdot X+k)^2$)
$(H^{D-1}, E^{0,1})$:	$\sigma = 0$	I-A, (& I-C for $K=-1$)
	$\sigma \neq 0$	I-C ($U \not\propto (P\cdot Y+k)^2$)

metric must take the form (4·11) in this case, which is excluded in the definition of the type II. Therefore, all the type II solutions are static, and are represented as $F(x) = 0$ in terms of a function on the base space M_λ^{D-1}. This implies that the brane projects to an everywhere umbilical hypersurface with $K_{ab} = \sigma g_{ab}$ in the base space. In this section, using this fact, we show that brane configurations in type II geometries belong to the type I, unless $U(R)$ takes some special form, and classify possible configurations instead of using the explicit solutions obtained in §3, because it is much simpler. The main result is summarized in Table II.

The general scheme of the argument is as follows. First, we relate a coordinate system in which the metric of M_λ^{D-1} is written in the decomposed form (5·19) or (5·21) and a global homogeneous coordinate system for M_λ^{D-1} in which the action of the isometry group has a simple representation, as was done in §5.2. Then, since both the argument of U and the function $F(x)$ defining the brane Σ^{D-1} are obtained from a v-coordinate giving the decomposition (5·18) through isometries of M_λ^{D-1}, we can easily find possible canonical forms for the argument of U and $F(x)$. Next, we determine U by the condition $K_{TT} = \sigma g_{TT}$, which is simply written

$$\pm\frac{\nabla F\cdot\nabla U}{2U|\nabla F|} = \sigma \tag{5·42}$$

in the present case, where the sign on the left-hand side must be chosen so that the normal vector $\pm\nabla F/|\nabla F|$ gives the correct sign in $K_{ab} = \sigma g_{ab}$.

5.4.1. $M^D = (E^{D-1}, E^{0,1})$

In this case, the bulk metric is written

$$ds_D^2 = d\boldsymbol{x}^2 - U dT^2. \tag{5·43}$$

Since $\lambda = 0$, the decomposition (5·18) gives $K = 0$ or $K = 1$, for which v in (5·21) or r in (5·19) depends only on x^{D-1} or $\boldsymbol{x}^2$, respectively, after some isometric transformations. Hence, we can assume that U is a function of x^{D-1} or $\boldsymbol{x}^2$, and the generic form of a static brane configuration is

$$\sigma = 0 : F(\boldsymbol{x}) \equiv \boldsymbol{p}\cdot\boldsymbol{x} - A = 0, \tag{5·44a}$$
$$\sigma \neq 0 : F(\boldsymbol{x}) \equiv (\boldsymbol{x}-\boldsymbol{a})^2 - 1/\sigma^2 = 0, \tag{5·44b}$$

where $\boldsymbol{p}$ and $\boldsymbol{a}$ are $(D-1)$-dimensional constant vectors and A is a constant.

We first consider the case $U = U(x^{D-1})$. In this case, for a brane with $\sigma = 0$, F can be put into the form $F = qx^{D-2} + px^{D-1} - A$ by an isometry of E^{D-1} preserving U. Then, (5·42) reduces to $pU'(x^{D-1}) = 0$. If $p \neq 0$, we obtain $U' = 0$. Hence, if $p \neq 0$ and $q \neq 0$, U must be constant, and the bulk metric (5·43) represents a flat metric, which implies that the solution belongs to the type IV. On the other hand, if $p \neq 0$ and $q = 0$, U can be arbitrary, and the brane configuration is given by a solution to $U'(x^{D-1}) = 0$. Hence, the solution belongs to the type I-C. U can be arbitrary as well for $p = 0$, but in this case, the brane configuration is isometric to $x^{D-2} = 0$. It is easy to see that this solution belongs to the type I-A when expressed in terms of appropriate spherical coordinates for E^{D-1}.

Next, for $\sigma \neq 0$, (5·42) with the brane configuration (5·44b) becomes $U'/U = 2(x^{D-1} - a^{D-1})$. Hence, U can be put into the form $U = (x^{D-1})^2$ by an isometry and a constant rescaling of T. Then, the bulk metric takes the Rindler form of the flat metric, and the solution belongs to the type IV.

These results for $U = U(x^{D-1})$ justify the exclusion of the bulk geometry represented as $ds_D^2 = -U(R)dT^2 + dR^2 + d\boldsymbol{z}^2$ from the type II.

For the case $U = U(\boldsymbol{x}^2)$, (5·42) for a brane with $\sigma = 0$ is equivalent to $AU' = 0$ on the brane. Hence, when $A = 0$, U is arbitrary, and the brane configuration is isometric to $x^{D-2} = x^{D-1}$. This solution belongs to the type I-A. On the other hand, when $A \neq 0$, we obtain the constraint U =const for $\boldsymbol{p}^2\boldsymbol{x}^2 \geq A^2$. If U is an analytic function, this solution belongs to the type IV.

Next, for a brane with $\sigma \neq 0$, (5·42) with (5·44b) reads

$$\frac{U'}{U} = \frac{2}{1/\sigma^2 - \boldsymbol{a}^2 + \boldsymbol{x}^2}. \tag{5·45}$$

For $\boldsymbol{a} \neq 0$, this determines U to be proportional to (3·61) with $R = |\boldsymbol{x}|$, and the brane configuration cannot have a type I representation. In contrast, for $\boldsymbol{a} = 0$, this equation gives an equation for σ^2, and its solution (if one exists) belongs to the type I-C.

To summarize, if the metric (5·43) does not have a constant curvature and U is not of the form $U(\boldsymbol{p} \cdot \boldsymbol{x})$, a brane with $\sigma = 0$ always has static configurations of the type I-A, which are mutually isometric. If U is an analytic function, these exhaust all possible configurations of a brane with $\sigma = 0$. Similarly, a brane with $\sigma \neq 0$ is always static in the type II bulk spacetime. Further, except for exceptional configurations of the type I-C for an $O(D-1)$-invariant U, a brane with $\sigma \neq 0$ can be embedded in the type II geometry only if U has the special form

$$U = (\boldsymbol{x}^2 + k)^2. \tag{5·46}$$

Brane configurations in this geometry are given by (5·44b) with

$$\boldsymbol{a}^2 = \frac{1}{\sigma^2} - k, \tag{5·47}$$

and do not have a type I representation. Here, note that, strictly speaking, U has

to take this form only in the region

$$(|\boldsymbol{a}| - 1/|\sigma|)^2 \leq \boldsymbol{x}^2 \leq (|\boldsymbol{a}| + 1/|\sigma|)^2, \tag{5·48}$$

unless the analyticity of U is assumed.

5.4.2. $M^D = (S^{D-1}, E^{0,1})$

In this case, the bulk metric is written

$$ds_D^2 = \ell^2 dX \cdot dX - U dT^2;\ X \cdot X \equiv (X^1)^2 + \cdots + (X^D)^2 = 1. \tag{5·49}$$

Since $\lambda > 0$, the decomposition (5·18) gives $K = 1$, and v =const surfaces are given by the intersections of parallel hyperplanes and the sphere $X \cdot X = 1$ in the X-space. Hence, with an appropriate choice of the X-coordinates, we can assume that U depends only on X^D and the brane is represented as

$$P \cdot X = \ell\sigma;\ P \cdot P = 1 + \ell^2\sigma^2. \tag{5·50}$$

For this choice, since X^M $(M = 1, \cdots, D)$ satisfies

$$\nabla X^M \cdot \nabla X^N = \delta^{MN} - X^M X^N, \tag{5·51}$$

where ∇ is the covariant derivative of the unit $(D-1)$-dimensional sphere, (5·42) gives

$$(P^D - \ell\sigma X^D)\frac{U'}{U} = -2\ell\sigma. \tag{5·52}$$

First, for a brane with $\sigma = 0$, this equation becomes trivial for configurations with $P^D = 0$, which belongs to the type I-A. In contrast, for $P^D \neq 0$ and $P^a = 0$ $(a = 1, \cdots, D-1)$, this equation reads $U'(0) = 0$, and if this condition is satisfied, there exists a configuration of the type I-C. Finally, configurations with $P^D \neq 0$ and $(P^a) \neq 0$ are allowed only when U is constant. This is the only special case in which $(S^{D-1}, E^{0,1})$ has an extra continuous symmetry, as shown in Table III. In this special geometry, configurations of the type I-C are isometric to those of the type I-A.

Next, for a brane with $\sigma \neq 0$ and $P^a = 0$, (5·52) is equivalent to the equation for the location of a type I-C brane, while for a brane with $\sigma \neq 0$ and $(P^a) \neq 0$, it requires U to be proportional to $(P^D - \ell\sigma X^D)^2$. Hence, the bulk spacetime with the structure $(S^{D-1}, E^{0,1})$ can contain a brane with $\sigma \neq 0$ that is not of the type I-C only when U has the form

$$U = (X^D + k)^2, \tag{5·53}$$

where $k \neq 0$, because the spacetime becomes dS^D for $U = (X^D)^2$. The corresponding brane configuration is written

$$-kX^D + p_a X^a = 1;\ k^2 + \boldsymbol{p}^2 = 1 + \frac{1}{\ell^2\sigma^2}, \tag{5·54}$$

which does not have a type I representation. It is easy to see that this can be expressed in the form of (3·58) with $K = 1$ and $\lambda = 1/\ell^2$ if we set $R = \ell\sqrt{1-(X^D)^2}$ and $X = \mp a\ell\sigma p_a X^a/\sqrt{1-(X^D)^2}$, because $X = b_a \Omega^a$ satisfies (3·42) with $K = 1$ and $c = b_a b^a$ on the unit sphere $\Omega_a \Omega^a = 1$, as shown in Appendix C.

5.4.3. $M^D = (H^{D-1}, E^{0,1})$

In this case, the bulk metric is written

$$ds_D^2 = \ell^2 dY \cdot dY - U dT^2; \ Y \cdot Y \equiv -(Y^0)^2 + (Y^1)^2 + \cdots + (Y^{D-1})^2 = -1, \ Y^0 \geq 1. \tag{5·55}$$

Now, the hyperbolic space H^{D-1} has tree types of slicing by constant curvature hypersurfaces: The decomposition of the form (5·19) with $K = 0, +1$ and -1 is obtained for $r = \ell(Y^0 - Y^{D-1}), \ell\sqrt{(Y^0)^2 - 1}$ and $\ell\sqrt{(Y^{D-1})^2 + 1}$, respectively. Since a generic decomposition is isometric to one of these, the level surfaces of r are always given by the intersections of hyperplanes $P \cdot Y$ =const with the hyperboloid $Y \cdot Y = -1$ in the Y-space, and the sign of K is the same as that of $-P \cdot P$. In particular, the generic configuration of a brane is expressed as

$$P \cdot Y = A; \quad P \cdot P = 1 - \ell^2 \sigma^2, \tag{5·56}$$

where

$$A = \ell\sigma \ (\ell^2\sigma^2 \neq 1), \quad A \neq 0 \ (\ell^2\sigma^2 = 1). \tag{5·57}$$

Let us first treat the case $U = U(Y^{D-1})$. Since Y^M $(M = 0, \cdots, D-1)$ as a function on H^{D-1} satisfies

$$\nabla Y^M \cdot \nabla Y^N = \eta^{MN} + Y^M Y^N, \tag{5·58}$$

where ∇ is the covariant derivative for the unit hyperboloid $Y \cdot Y = -1$, (5·42) reads

$$(P^{D-1} + AY^{D-1})\frac{U'}{U} = 2\ell\sigma\sqrt{1 - \ell^2\sigma^2 + A^2}. \tag{5·59}$$

First, for a brane with $\sigma = 0$, $A = 0$ from (5·57) and the left-hand side of this equation must vanish. When $P^{D-1} = 0$, U can be arbitrary, and the brane configuration can be transformed into $Y^{D-3} = Y^{D-2}$. This gives a type I-A solution when it is expressed in appropriate spherical coordinates for Y^M. In contrast, for $P^{D-1} \neq 0$ and $P^\mu = 0$ $(\mu = 0, \cdots, D-2)$, (5·57) reduces to $U'(0) = 0$, and if this equation is satisfied, there exists a type I-C solution. Finally, configurations with $P^{D-1} \neq 0$ and $(P^\mu) \neq 0$ are possible only when U is constant. This is the only special case in which the bulk spacetime $(H^{D-1}, E^{0,1})$ has an additional continuous symmetry, as shown in Table III. In this special case, all brane configurations with $\sigma = 0$ are mutually isometric and have a type I-A representation.

Next, for $\sigma \neq 0$, A does not vanish, as seen from (5·57), and the right-hand side of (5·59) is a non-vanishing constant. Hence, if $(P^\mu) \neq 0$, U must be proportional to $(P^{D-1} + AY^{D-1})^2$, and by a constant rescaling of T, it can be written

$$U = (Y^{D-1} + k)^2, \tag{5·60}$$

where $k \neq 0$, because the bulk spacetime becomes AdS^D for $U = (Y^{D-1})^2$. The corresponding brane configuration is expressed as

$$kY^{D-1} + p_\mu Y^\mu = 1; \quad k^2 + p_\mu p^\mu = \frac{1}{\ell^2\sigma^2} - 1, \tag{5·61}$$

which does not have a type I representation. This corresponds to the solution (3·58) with $K=-1$ and $\lambda=-1/\ell^2$ for $R=\ell\sqrt{(Y^{D-1})^2+1}$ and $X=\pm a\ell^2\sigma p_\mu Y^\mu/R$, because $X=b_\mu\bar{Y}^\mu$ satisfies (3·42) with $K=-1$ and $c=b_\mu b^\mu$ on the unit hyperboloid $\bar{Y}\cdot\bar{Y}=-1$, as shown in Appendix C. If $k=0$, this solution belongs to the type IV. On the other hand, if $P^\mu=0$, we obtain a solution belonging to the type I-C.

Next, let us consider the case $U=U(Y^0-Y^{D-1})$. In this case, (5·42) reads

$$[P^0-P^{D-1}+A(Y^0-Y^{D-1})]\frac{U'}{U}=2\ell\sigma\sqrt{1-\ell^2\sigma^2+A^2}. \tag{5·62}$$

For $\sigma=0$, we can conclude that this equation becomes trivial for type I-A configurations, and there exists a type I-C configuration if $U'(0)=0$, by the argument given in the previous case. For $\sigma\neq 0$, if the brane configuration is not of the type I-C, (5·62) constrains U as

$$U=(Y^0-Y^{D-1}+k)^2, \tag{5·63}$$

where $M^D=\mathrm{AdS}^D$ if $k=0$. The corresponding brane configuration is expressed as

$$-k(Y^0+Y^{D-1})+q(Y^0-Y^{D-1})+2p_iY^i=2;\quad kq+p_ip^i=\frac{1}{\ell^2\sigma^2}-1, \tag{5·64}$$

where q and p_i $(i=1,\cdots,D-2)$ are constants, and does not have a type I representation. By a Lorentz transformation of Y and a rescaling of U, this brane configuration can be put into the canonical form

$$\left(2-\frac{1}{\ell^2\sigma^2}\right)Y^0+\frac{1}{\ell^2\sigma^2}Y^{D-1}=-2k;\quad k=\pm 1, \tag{5·65}$$

which can be further rewritten in the form (3·54) with $\lambda=-1/\ell^2$, if we introduce the coordinates $(R,\boldsymbol{z})$ as $R=\ell|Y^0-Y^{D-1}|$ and $z^i=Y^i/(Y^0-Y^{D-1})$.

Finally, we consider the case $U=U(Y^0)$. In this case, (5·42) reads

$$(P^0+AY^0)\frac{U'}{U}=2\ell\sigma\sqrt{1-\ell^2\sigma^2+A^2}. \tag{5·66}$$

For $\sigma=0$, we can again conclude that this equation becomes trivial for type I-A configurations, and there exists a type I-C configuration if $U'(0)=0$. Further, if there exists a brane with $\sigma\neq 0$ that is not of the type I-C, (5·66) leads to the solution

$$U=(Y^0+k)^2, \tag{5·67}$$

where the spacetime again becomes AdS^D for $k=0$. The corresponding brane configuration is expressed as

$$-kY^0+p_aY^a=1;\quad p_ap^a=k^2-1+\frac{1}{\ell^2\sigma^2}, \tag{5·68}$$

where a runs over $1,\cdots,D-1$, and does not have a type I representation. This configuration can be put into the form (3·58) with $\lambda=-1/\ell^2$, $K=1$ and $X=\pm a\ell\sigma p_a\Omega^a$ in the coordinates $R=\ell\sqrt{(Y^0)^2-1}$ and $\Omega^a=\ell Y^a/R$.

5.5. *Relations among the types*

By the arguments up to this point, it is clear that the types III and IV are subclasses of the type I, and solutions of the type II also belong to the type I, unless $\sigma \neq 0$ and U is proportional to (5·46), (5·53), (5·60), (5·63) or (5·67) in some homogeneous coordinate system for the constant curvature space M_λ^{D-1}. In this section, we show that the type-II solutions with $\sigma \neq 0$ for these forms of U really do not belong to the type I. We also clarify relations among the bulk geometries of the types I-B, II, III and IV.

5.5.1. The type II vs the types I-B and IV

We can show that the type II is mutually exclusive with the type IV by calculating the Riemann tensor for the metric (5·6). First, note that when this metric is put into the form (2·9), we obtain $(K, W, S) = (0, 1, 1)$ and $(1, 1, R)$ for $\lambda = 0$, and $W = K - \lambda R^2$ and $S = R$ for $\lambda \neq 0$. Inserting these into (A·4), we find

$$\tilde{R}_{ijkl} = \lambda S^4(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}), \tag{5·69}$$

$$\tilde{R}^i_{RjR} = \frac{\lambda}{W}\delta^i_j, \tag{5·70}$$

$$\tilde{R}^i_{TjT} = \frac{WU'S'}{2S}\delta^i_j. \tag{5·71}$$

Hence, if the spacetime has a constant curvature, $WU'S'/(2US)$ must be equal to $-\lambda$. This requires that U be proportional to W, except in the case $K = 0$ and $W = S = 1$. For this exceptional case, the condition ${}^2R = 2\lambda$ requires that U be proportional to $(aR+b)^2$, which can be set to 1 or R^2 by a constant shift of R and a rescaling of T. This result, with the argument in §5.4, shows that the type I-B and the type II are also mutually exclusive.

5.5.2. The type III vs the types I-B, II and IV

The metric (5·7) corresponds to (2·9) with $R = v$, $U = f^2(v)$, $W = 1$, $S = f(v)$ and $K = 0$. Hence, from (A·4), we obtain

$$\tilde{R}_{ijkl} = -f^2(f')^2(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}). \tag{5·72}$$

From this, it follows that the spacetime has a constant curvature only when $f =$const or $f = f_0 e^{v/\ell}$, which correspond to $E^{D-1,1}$ and $\mathrm{AdS}^D(\ell)$, respectively.

This results also implies that the type III geometry cannot be of the type II for the choice of the time coordinate T in terms of which the metric is written in the form (5·7). Hence, if we can show that all time-like Killing vectors preserve the metric $\eta_{\mu\nu}dx^\mu dx^\nu$ of the fibre $E^{D-2,1}$, the mutual exclusiveness of the type III and the type II geometries is concluded.

Let $\xi = \xi^v\partial_v + \xi^\mu\partial_\mu$ be a Killing vector of (5·7). Then, since the non-vanishing components of the Christoffel symbols are given by

$$\mathit{\Gamma}^v_{\mu\nu} = -ff'\eta_{\mu\nu},\ \mathit{\Gamma}^\mu_{v\nu} = \frac{f'}{f}\delta^\mu_\nu, \tag{5·73}$$

the Killing equation yields the three equations

$$\partial_v\xi_v = 0, \tag{5·74a}$$

Table III. A list of geometries with higher symmetries

Type	Isometry	Metric
Type I		
$E^{1,1}\times M_K^{D-2}$	$IO(1,1)\times G(D-2,K)$	$ds^2=-dT^2+dr^2+\ell^2 d\sigma_K^2$
$(K=\pm1)$	ibid	$ds^2=-r^2dT^2+dr^2+\ell^2 d\sigma_K^2$
$\mathrm{dS}^2(1/\mu)\times M_K^{D-2}$	$O(2,1)\times G(D-2,K)$	$ds^2=-(1-\mu^2R^2)dT^2+\frac{dR^2}{1-\mu^2R^2}+\ell^2 d\sigma_K^2$
$\mathrm{AdS}^2(1/\mu)\times M_K^{D-2}$	$O(1,2)\times G(D-2,K)$	$ds^2=-(K'+\mu^2R^2)dT^2+\frac{dR^2}{K'+\mu^2R^2}+\ell^2 d\sigma_K^2$
$(M^{1,1},E^{D-2})$	$\mathbb{R}_+\times IO(1)\times IO(D-2)$	$ds^2=-\left(\frac{R}{\ell}\right)^{2n}dT^2+\frac{\ell^2}{R^2}dR^2+R^2d\boldsymbol{x}^2\ (n\neq1)$
Type II		
$E^{0,1}\times M_\lambda^{D-1}$	$IO(1)\times G(D-1,\lambda)$	$ds^2=-dT^2+\frac{dR^2}{K-\lambda R^2}+R^2 d\sigma_K^2;\ (\lambda\neq0)$
Type III		
$(E^1,E^{D-2,1})$	$IO(D-2,1)$	$ds^2=dv^2+f(v)^2\eta_{\mu\nu}dx^\mu dx^\nu;\ \left(\frac{f'}{f}\right)'\neq0$
Type IV		
$E^{D-1,1}$	$IO(D-1,1)$	$ds^2=-dT^2+dr^2+d\boldsymbol{x}^2$,
		$ds^2=-dT^2+dR^2+R^2d\Omega_{D-2}^2$,
		$ds^2=-r^2dT^2+dr^2+d\boldsymbol{x}^2$
$\mathrm{dS}^D(\ell)$	$O(D,1)$	$ds^2=-(1-R^2/\ell^2)dT^2+\frac{dR^2}{1-R^2/\ell^2}+R^2d\Omega_{D-2}^2$
$\mathrm{AdS}^D(\ell)$	$O(D-1,2)$	$ds^2=-(K+R^2/\ell^2)dT^2+\frac{dR^2}{K+R^2/\ell^2}+R^2d\sigma_K^2$

$$\partial_v(f^{-2}\xi_\mu)+f^{-2}\partial_\mu\xi_v=0, \tag{5·74b}$$
$$\partial_\mu\xi_\nu+\partial_\nu\xi_\mu+2ff'\xi_v\eta_{\mu\nu}=0. \tag{5·74c}$$

The first equation implies that ξ_v depends only on x^μ, and the integration of the second equation gives

$$f^{-2}\xi_\mu=-\partial_\mu\xi_v\int f^{-2}dv+\eta_\mu(x), \tag{5·75}$$

where $\eta_\mu(x)$ is a field that is independent of v. Putting this into the third equation, we obtain

$$\partial_\mu\eta_\nu+\partial_\nu\eta_\mu-2\partial_\mu\partial_\nu\xi_v\int f^{-2}dv+2\frac{f'}{f}\xi_v\eta_{\mu\nu}=0. \tag{5·76}$$

If $1/f^2$ and $(f'/f)'$ are linearly independent, it follows from (5·76) that $\xi_v\equiv 0$, which implies that the Killing vector preserves the fibre metric. Therefore, let us assume that $(f'/f)'=a/f^2$, where a is a non-vanishing constant because the spacetime is not of the type IV. Then, f can be written $f^2=1/(ar^2+2br+c)$ in terms of the variable $r=\int dv/f^2$. Inserting this into (5·76), we obtain $\partial_\mu\partial_\nu\xi_v+a\xi_v=0$. It follows from this equation that $a\partial_\mu\xi_v=0$, and hence $\xi_v=0$. Therefore, all time-like Killing vectors of the type III geometry preserve the fibre metric. This result also implies that the type III geometry is mutually exclusive with the type I-B geometry.

5.5.3. Types I-A, I-B and I-C

We have shown that the bulk geometries of the types I-B, II and III as well as the types II, III and IV are never mutually isometric. We have also clarified the

relations between the type I and the types II, III and IV. Therefore, we can complete the geometrical classification of solutions if we clarify the uniqueness of the type I representations and the relations among the types I-A, I-B and I-C in the case in which the bulk geometry does not belong to the type II, III or IV. For this purpose, we use Table III, which lists all higher symmetries that an $IO(1) \times G(D-2, K)$-invariant spacetime can have. This list wasobtained by classifying all possible solutions to the Killingequation, but we do not give its derivation here, because it is ratherlengthy and tedious. In this table, the terms "type II", "typeIII" and "type IV" have the meanings defined in the beginning ofthis section, while the term "type I" refers to all geometries other than those of these three types.

First, note that $G(D-2, \pm 1)$ are semi-simple and do not have a non-trivial normal subgroup. In contrast, $G(D-2, 0) = IO(D-2)$ has the normal subgroup $\mathbb{R}^{D-2}$, but the latter cannot be a normal subgroup of $IO(1,1)$ or $O(2,1) = O(1,2)$ for $D \geq 4$. Hence, from Table III we see that a subgroup of the type $G(D-2, K)$ contained in the isometry group of a spacetime that does not belong to the type II, III or IV is unique. This implies that brane configurations of the type I-A are mutually isometric and do not have a representation of the type I-B or I-C if the bulk geometry does not belong to the type II, III or IV. Further, it is also easy to see that configurations of the types I-B and I-C become mutually isometric only in the case in which the bulk spacetime has the structure $M_\lambda^{1,1} \times M_K^{D-2}$.

Next, we comment on consequences for the other types obtained from Table III. First, the isometry group of a type III geometry is always $IO(D-2, 1)$. Hence, a configuration of the type I-C, if it exists, is not isometric to configurations of the type I-A. For the type IV, all configurations of a brane with the same σ^2 are mutually isometric. Finally, for a type II geometry, configurations of the types I-A and I-C are connected by isometries only when U is constant, for which all configurations of the type I are mutually isometric.

§6. Summary and discussion

In this paper, we have classified completely all possible configurations of a brane, that is, a time-like hypersurface satisfying the condition $K_{\mu\nu} = \sigma g_{\mu\nu}$ in static D-dimensional spacetimes ($D \geq 4$) with spatial symmetry $G(D-2, K) = IO(D-2)(K=0)$, $O(D-1)(K=1)$ or $O_+(D-2,1)(K=-1)$, which has the bundle structure (B^2, M_K^{D-2}) with a 2-dimensional base spacetime B^2 and a fibre M_K^{D-2} with a constant curvature K. We summarize the main results in the following two theorems.

Theorem 1 *Configurations of a brane with $\sigma = 0$ and allowed geometries are classified into the following three types:*

I-A) Brane configurations that are represented by subbundles $\Sigma = (B^2, F)$ of $M^D = (B^2, M_K^{D-2})$, where F is a totally geodesic hypersurface M. Configurations of this type exist for any choice of U, W and S, and are mutually isometric. Each configuration is invariant under $IO(1) \times G(D-3, K')$ for some $K' \geq K$.

I-B) $G(D-2, K)$-invariant configurations which are represented as $R = R(T)$ by

solutions to

$$R_T^2 = WU(1-AU) \neq 0;\ \ A>0. \tag{6·1}$$

In this case, the bulk geometry is restricted to a simple product $B^2 \times M_K^{D-2}$*, for which* $S=$*const.*

I-C) *Static configurations expressed as* $R=$*const in terms of solutions to*

$$U'=0, \quad S'=0. \tag{6·2}$$

Each configuration of this type corresponds to an $IO(1)\times G(D-2,K)$*-invariant totally geodesic hypersurface.*

A brane with $\sigma=0$ *can only have configurations of the type I-A in* AdS^D*, while it can also have configurations of the types I-B and I-C in* $E^{D-1,1}$ *and of the type I-B in* dS^D*. The latter are all mutually isometric in a given bulk geometry. In a bulk spacetime that does not have constant curvature, configurations of the type I-B or I-C become isometric to those of the type I-A only when the bulk spacetime has the product structure* $E^{0,1}\times M_K^{D-1}$*.*

Since the condition $\sigma=0$ is equivalent to the condition that the hypersurface is totally geodesic, this theorem gives the complete classification of totally geodesic time-like hypersurfaces in spacetimes with the $IO(1)\times G(D-2,K)$ symmetry. From this point of view, the universal existence of configurations of the type I-A is rather trivial, because every totally geodesic hypersurface of a constant curvature space $R=$const is a fixed-point set for some involutive isometry that is also an isometry of the whole bulk spacetime. However, the $G(D-2,K)$ invariance of all the other configurations is a non-trivial result.

Theorem 2 *A brane with* $\sigma\neq 0$ *can exist only for special bulk geometries. These bulk geometries and corresponding brane configurations are classified into the following three types:*

I-B) *Brane configurations that are* $G(D-2,K)$*-invariant and represented as* $R=R(T)$ *by solutions to*

$$R_T^2 = U^2\left(1-\frac{U}{\sigma^2 R^2}\right) \not\equiv 0. \tag{6·3}$$

In this case, the bulk geometry is restricted to those with $U=W$ *and* $S=R$*. For* $S=R$*, the condition* $U=W$ *is equivalent to the condition that the Ricci tensor is isotropic in planes orthogonal to* $G(D-2,K)$*-orbits.*

I-C) *Static and* $G(D-2,K)$*-invariant brane configurations expressed as* $R=$*const in terms of solutions to*

$$\frac{U'}{U}=\frac{2S'}{S}, \quad W\left(\frac{S'}{S}\right)^2=\sigma^2. \tag{6·4}$$

II) *Static brane configurations in the bulk geometries with metrics of the form*

$$ds_D^2 = d\sigma_{\lambda,D-1}^2 - UdT^2, \tag{6·5}$$

where $d\sigma^2_{\lambda,D-1}$ *is the metric of a* $(D-1)$*-dimensional constant curvature space* M^{D-1}_λ *with sectional curvature* λ*. In this case,* U *is a function on this space that is invariant under a subgroup* $G(D-2,K)$ *of the isometry group of* M^{D-1}_λ*. Allowed forms of* U *and brane configurations are given as follows:*

i) $\lambda = 0$*: In a Cartesian coordinate system* $\boldsymbol{x}$ *for* E^{D-1}*,* $U = ((\boldsymbol{x}-\boldsymbol{a})^2 + k)^2$ *and brane configurations are represented as* $(\boldsymbol{x}-\boldsymbol{b})^2 = 1/\sigma^2$ *with* $(\boldsymbol{b}-\boldsymbol{a})^2 = 1/\sigma^2 - k$*.*

ii) $\lambda = 1/\ell^2 > 0$*: In a homogeneous coordinate system* X *in which* S^{D-1} *is expressed as* $X \cdot X = 1$*,* $U = (P \cdot X + k)^2$ *(*$k \neq 0$*) and brane configurations are represented as* $Q \cdot X = 1$ *with* $Q \cdot P = -k$ *and* $Q \cdot Q = 1 + 1/(\ell\sigma)^2$*.*

iii) $\lambda = -1/\ell^2 < 0$*: In a homogeneous coordinate system* Y *in which* H^{D-1} *is expressed as* $Y \cdot Y = -1$*,* $U = (P \cdot Y + k)^2$ *(*$k \neq 0$*) and brane configurations are represented as* $Q \cdot Y = 1$ *with* $Q \cdot P = k$ *and* $Q \cdot Q = 1/(\ell\sigma)^2 - 1$*.*

Although these brane configurations are not $G(D-2,K)$*-invariant, they are still* $G(D-3,K')$*-invariant for some* $K' \geq K$ *and mutually isometric.*

Configurations of the type I-B and those of the type I-C are isometric only when the bulk spacetime is a product of a 2-dimensional constant curvature spacetime and a constant curvature space M^{D-2}_K*, provided that it is not a constant curvature spacetime. In constant curvature spacetimes, a brane with* $\sigma \neq 0$ *can have only configurations of the type I-B, except for a brane with* $\ell^2\sigma^2 = 1$ *in* $AdS^D(\ell)$*, for which only 1-C configurations are allowed.*

Here, note that type I-B configurations can have different isometry classes only when a type I-C configuration exists. Since a type I-C configuration does not exist in a generic spacetime, this implies that configurations of a brane with $\sigma \neq 0$ are mutually isometric in a generic type I-B geometry.

Now we give some comments on implications of these results. First, as stated in the Introduction, one purpose of the present paper has been to extend the analysis done by Chamblin, Hawking and Reall on the possibility of a black hole geometry being induced on a vacuum brane from a (pseudo-)spherically symmetric bulk spacetime. From this point of view, only configurations of a brane with spatial symmetry lower than that of the bulk are relevant. However, our results show that such configurations are static and possible only in bulk spacetimes of the type II with the special forms of U given above. It is clear that these solutions do not give a black hole geometry on the brane, because the space has a constant curvature for them. Further, although U can vanish when k has the appropriate sign, the spacetime curvature diverges at points where U vanishes. Hence, we can conclude that we can never obtain a brane configuration representing a vacuum black hole from a static spacetime solution to the Einstein equations with the assumed spatial symmetry, irrespective of the bulk matter content.

Next, we comment on mathematical consequences. First, our results extend the theorem concerning everywhere umbilical hypersurfaces in Euclidean spaces mentioned in the Introduction to the case of constant curvature spacetimes. Our results also extend this theorem to non-constant curvature spacetimes, in the sense that spa-

tial sections of a brane with $\sigma \neq 0$ always respect the spatial symmetry $G(D-2,K)$ of a spacetime, and a brane configuration is always $G(D-3,K')$-invariant with $K' \geq K$. Our results also provide extensions of the famous rigidity theorem on hypersurfaces in Euclidean spaces, which states that two embeddings of a Riemannian manifold into a Euclidean space as hypersurfaces become isometric if the extrinsic curvatures coincide.[28] For example, our results on a brane in constant curvature spacetimes give a direct extension of this theorem to time-like hypersurfaces with $K_{\mu\nu} = \sigma g_{\mu\nu}$ in constant curvature spacetimes. Further, our results imply that the non-existence of a type I-C solution is a necessary condition for the same rigidity to hold for hypersurfaces with $\sigma \neq 0$ in spacetimes that do not have a constant curvature.

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Appendix A
—— *Geometric Quantities for the Bulk Geometry* ——

In this appendix we list formulas for the Christoffel symbols and the curvature tensor of the bulk metric (2·9).

For the bulk metric (2·9), the non-vanishing components of the Christoffel symbols are given by

$$\tilde{\Gamma}^T_{RT} = \frac{U'}{2U}, \tag{A·1a}$$

$$\tilde{\Gamma}^R_{TT} = \frac{1}{2}WU',\ \tilde{\Gamma}^R_{RR} = -\frac{W'}{2W},\ \tilde{\Gamma}^R_{ij} = -SS'W\gamma_{ij}, \tag{A·1b}$$

$$\tilde{\Gamma}^i_{jR} = \frac{S'}{S}\delta^i_j,\ \tilde{\Gamma}^i_{jk} = \Gamma^i_{jk}, \tag{A·1c}$$

where the prime denotes differentiation with respect to the argument of the corresponding function, and Γ^i_{jk} is the Christoffel symbol for γ_{ij}, which is expressed in terms of $\rho(\chi)$ and the Christoffel symbol $\hat{\Gamma}^A_{BC}$ for the metric $\hat{\gamma}_{AB}$ as

$$\Gamma^\chi_{AB} = -\rho\rho'\hat{\gamma}_{AB},\ \Gamma^A_{B\chi} = \frac{\rho'}{\rho}\delta^A_B,\ \Gamma^A_{BC} = \hat{\Gamma}^A_{BC}. \tag{A·2}$$

The curvature tensors have simpler expressions when we write the bulk metric as

$$ds^2_D = g_{ab}dx^a dx^b + S(R)^2\gamma_{ij}dz^i dz^j. \tag{A·3}$$

With this notation, the Riemann curvature tensor is given by

$$\tilde{R}_{abcd} = \frac{1}{2}{}^2R(g_{ac}g_{bd} - g_{ad}g_{bc}), \tag{A·4a}$$

$$\tilde{R}^i{}_{ajb} = \left(\frac{S'}{S}\tilde{\Gamma}^R_{ab} - \frac{S''}{S}\delta^R_a\delta^R_b\right)\delta^i_j, \tag{A·4b}$$

$$\tilde{R}_{ijkl} = (K - S'^2W)S^2(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}), \tag{A·4c}$$

and the Ricci tensor is given by

$$\tilde{R}_{ab} = \frac{1}{2}{}^2Rg_{ab} + (D-2)\left(\frac{S'}{S}\tilde{\Gamma}^R_{ab} - \frac{S''}{S}\delta^R_a\delta^R_b\right), \tag{A·5a}$$

$$\tilde{R}_{ai} = 0, \tag{A·5b}$$

$$\tilde{R}^i_j = \left[-\frac{(UWS'^2)'}{2USS'} + (D-3)\frac{K - S'^2W}{S^2}\right]\delta^i_j, \tag{A·5c}$$

$$\tilde{R} = {}^2R - (D-2)\frac{(UWS'^2)'}{USS'} + (D-2)(D-3)\frac{K-W}{S^2}, \tag{A·5d}$$

where

$${}^2R = -\frac{1}{2U'}\left(\frac{WU'^2}{U}\right)' = -\frac{W}{U}\left(U'' + \frac{W'U'}{2W} - \frac{(U')^2}{2U}\right). \tag{A·6}$$

Appendix B
—— $\partial_T\partial_i u = f(T,\boldsymbol{z})\partial_T u\partial_i u$ ——

In this appendix, we show that if $u(T,\boldsymbol{z})$ ($\boldsymbol{z} = (z^1,\cdots,z^n)$) satisfies the equation

$$\partial_T\partial_i u = f(T,\boldsymbol{z})\partial_T u\partial_i u, \tag{B·1}$$

u can be written as $u = F(T, X(\boldsymbol{z}))$ in terms of some function $X(\boldsymbol{z})$.

First, note that from this equation it immediately follows that for a fixed value of T, $\partial_T u$ is constant on a hypersurface in the z space on which u is constant. Hence, u_T can be written in terms of some function $G(T,u)$ as $\partial_T u = G(T,u)$. Let the solution of the ordinary differential equation $dv/dT = G(T,v)$ for the initial condition $v(T_0) = v_0$ be $v = F(T,v_0)$. Then, for any function $X(\boldsymbol{z})$, we obtain

$$\partial_T F(T,X(\boldsymbol{z})) = G(T, F(T,X(\boldsymbol{z}))), \tag{B·2}$$

and $F(T_0, X(\boldsymbol{z})) = X(\boldsymbol{z})$. Since the solution of the equation $\partial_T u = G(T,u)$ is uniquely determined if $u(T_0,\boldsymbol{z})$ is given, this implies that any solution u to the equation $\partial_T u = G(T,u)$ can be written $u = F(T,X(\boldsymbol{z}))$ with $X(\boldsymbol{z}) = u(T_0,\boldsymbol{z})$.

Appendix C
—— $D_iD_ju = \alpha g_{ij} + \beta D_iuD_ju$ ——

In this appendix, we give the general solution to the equation

$$D_iD_ju = \alpha g_{ij} + \beta D_iuD_ju \tag{C·1}$$

on an n-dimensional constant curvature space M^n_K with sectional curvature K. Here, D_i is the covariant derivative with respect to the metric g_{ij} of M^n_K, and α and β are functions of u. We assume that $n > 1$.

First, we show that the problem can be reduced to the case $\beta = 0$. Let us change the unknown function u to v by $u = f(v)$. Then, (C·1) is transformed into

$$f' D_i D_j v = \alpha g_{ij} + \left((f')^2 \beta - f'' \right) D_i v D_j v. \tag{C·2}$$

Hence, if f is so chosen as to satisfy the equation $f'' = \beta(f)(f')^2$, i.e.,

$$v = f^{-1}(u) = \int du \exp - \int^u \beta(u') du', \tag{C·3}$$

we obtain an equation of the type (C·1) with $\beta = 0$.

For $\beta = 0$, the consistency of (C·1) gives

$$D_i \alpha = D^j D_i D_j u = R_i^j D_j u + D_i \triangle u = R_i^j D_j u + n D_i \alpha. \tag{C·4}$$

Here, since M_K^n is a constant curvature space, its Ricci tensor R_{ij} is

$$R_{ij} = (n-1) K g_{ij}. \tag{C·5}$$

Hence, we obtain

$$D_i(\alpha + K u) = 0, \tag{C·6}$$

whose general solution is

$$\alpha(u) = -K u + \alpha_0. \tag{C·7}$$

If α does not have this form, (C·1) with $\beta = 0$ has no solution.

First, we consider the case $K = 0$, in which we can choose a Cartesian coordinate system z^i for which $g_{ij} = \delta_{ij}$. In this coordinate system, the equation can be easily integrated. The general solution is given by

$$u = \frac{1}{2} \alpha_0 z \cdot z + a_i z^i + b, \tag{C·8}$$

where a^i and b are constants.

Next, in the case $K = \pm 1$, by the replacement $u - K\alpha_0 \to u$, the problem is reduced to that of the case $\alpha = -K u$,

$$D_i D_j u = -K g_{ij} u. \tag{C·9}$$

Then, it follows from this equation that

$$D_i[(Du)^2 + K u^2] = 2(D_i D_j u + K u g_{ij}) D^j u = 0, \tag{C·10}$$

where $(Du)^2 = g^{ij} D_i u D_j u$. Hence,

$$(Du)^2 + K u^2 = c \text{ (constant)}. \tag{C·11}$$

Further, for any curve $z^i = z^i(t)$, (C·9) gives the ordinary differential equation

$$\dot{u} = \dot{z}^i u_i, \ \dot{u}_i = \Gamma^j_{ik} \dot{z}^k u_j - K u g_{ij} \dot{z}^j. \tag{C·12}$$

This implies that u is uniquely determined if the values of u and $u_i = D_i u$ at some point are specified. In particular, $(Du)^2$ can vanish only at isolated points, and each level set of u is a smooth hypersurface, except at such points, for any non-trivial solution u.

Now, let us calculate the extrinsic curvature K_{AB} of the $u = \text{const}$ surface Σ_u. Since the unit normal to Σ_u is given by $n_A = D_A u/|Du|$, from (C·9), K_{AB} can be expressed as

$$K_{AB} = -D_A n_B = \frac{Ku}{|Du|} g_{AB}, \tag{C·13}$$

which implies that Σ_u gives an everywhere umbilical hypersurface. Hence, from the argument in §5.1, u can be expressed as a function of v for an appropriate choice of v giving the decomposition (5·18).

Now, using this observation, we derive expressions for u in terms of homogeneous coordinate systems for M^n_K. First, for $K = 1$, the metric of the unit sphere S^n embedded in E^{n+1} can be expressed in terms of a homogeneous coordinate system Ω^a as

$$ds^2 = d\Omega \cdot d\Omega; \;\; \Omega \cdot \Omega = 1. \tag{C·14}$$

From the observation above and the argument in §5.4.2, u must be a function of $X = P \cdot \Omega$ for some fixed constant vector $P = (P^a)$. Hence, (C·9) is written

$$u_X D_i D_j X + u_{XX} D_i X D_j X = -g_{ij} u. \tag{C·15}$$

However, it is easily checked that each Ω^a satisfies (C·9) with $K = 1$. This implies that $u(X)$ satisfies $X u_X = u$ and $u_{XX} = 0$. Therefore, the general solution to (C·9) with $K = 1$ is

$$u = P \cdot \Omega; \;\; c = P \cdot P, \tag{C·16}$$

where we have used the relation (5·51) to calculate c. Through an appropriate transformation in $O(n+1)$, u can be written $u = \sqrt{c}\,\Omega^{n+1}$. If we introduce the angular coordinate χ by $\Omega^{n+1} = \cos\chi$, u is given by

$$u = \sqrt{c} \cos\chi. \tag{C·17}$$

The derivation of the corresponding formula for $K = -1$ is almost the same. The metric of the unit hyperboloid H^n embedded in $E^{n,1}$ is expressed as

$$ds^2 = dY \cdot dY; \;\; Y \cdot Y = -1, \tag{C·18}$$

where each homogeneous coordinate Y^μ satisfies (C·9). Hence, by the same reasoning as that applied in the case $K = 1$, we can conclude that the general solution to (C·9) with $K = -1$ is

$$u = P \cdot Y; \;\; c = P \cdot P, \tag{C·19}$$

where we have used the relation (5·58) to calculate c.

We can also express this solution in terms of the coordinate system $(\chi, \theta, \cdots)$ used in (2·10), which is related to the homogeneous coordinate system Y by

$$Y^0 = \cosh\chi,\ Y^i = \sinh\chi\,\Omega^i.\ (i = 1, \cdots, n) \tag{C·20}$$

The result depends on the sign of c. First, for $c < 0$, we can put (C·19) into the form $u = \sqrt{|c|}Y^0$ by a transformation in $O_+(n, 1)$. Then, u is given by

$$u = \sqrt{|c|}\cosh\chi. \tag{C·21}$$

Second, for $c = 0$, we can put (C·19) into the form $u = k(Y^0 - Y^n)$ by a Lorentz transformation. Hence, u can be expressed in terms of χ and θ as

$$u = k(\cosh\chi - \sinh\chi\cos\theta). \tag{C·22}$$

Finally, for $c > 0$, we can put (C·19) into the form $u = \sqrt{c}Y^n$, which reads

$$u = \sqrt{c}\sinh\chi\cos\theta. \tag{C·23}$$

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